**FARVaR**: Functional Autoregressive Value-at-Risk

Charlie X. Cai, Minjoo Kim, Yongchoel Shin, and Qi Zhang

Current version: January 2015

Abstract

Motivated by the increasing availability and importance of intraday data in financial markets, we propose a functional autoregressive value-at-risk model (FARVaR) which utilizes the informational advantage of high-frequency intraday return distributions for forecasting daily value-at-risk. It is a semi-parametric model which has the advantage of modelling the density function nonparametrically while forecasting the density function through a functional autoregressive model. We show that FAR is the best predictor of the intraday density function and FARVaR produces the smallest value-at-risk forecast error. Furthermore, out-of-sample forecasting evaluations using various backtesting tools demonstrate that FARVaR overall performance is superior to those of both parametric and nonparametric value-at-risk models. In particular, it is shown to simultaneously strengthen the coverage ability, reduce the economic cost, and enhance the statistical adequacy.

Keywords: high-frequency intraday returns, functional autoregressive model, hybrid semiparametric approach, risk management, value-at-risk.

JEL Codes: C22; C51; C52; G11

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*Leeds University Business School, University of Leeds, Email: X.Cai@lubs.leeds.ac.uk.
†Corresponding author: Adam Smith Business School, University of Glasgow, Email: Minjoo.Kim@glasgow.ac.uk, Tel: +44 (0)141 330 7772, Fax: +44 (0)141 330 4939.
‡University of York, Email:yongcheol.shin@york.ac.uk.
§Leeds University Business School, University of Leeds, Email: busqz@leeds.ac.uk.
1 Introduction

After the recent financial crisis, the role of value-at-risk (VaR) in risk management has been subjected to intensive debates. While the crisis exposes its weaknesses such as ignoring liquidity and correlation risk, the general view in the industry is that VaR has been and will be one of the important and useful tools for risk management Croft (2011). To this end, the search for a more accurate VaR is continued and, importantly, it faces a new challenge.

The ever-increasing prominence of computer based trading in the financial market makes the study of risk arising from automated intraday activities an especially important issue. One important phenomenon of these activities is the increase in high-frequency trading (HFT). While the market microstructure literatures are still debating the impact of HFT on market qualities, one thing they can agree on is that intraday dynamics becomes increasingly important to the price discovery.¹ A more accurate VaR needs the ability to capture and analyse the information generated in these more frequent underlying data even for daily risk management. While the usefulness of the intraday data has been demonstrated extensively in the realized volatility literature (Andersen et al., 2001; Engle and Giampiero, 2006), the direct incorporation of intraday information in the risk modelling has been still premature (e.g. Fuertes and Olmo, 2013; Hallam and Olmo, 2014).

It is also well established that intraday returns follow a non-normal and time-varying distribution Andersen et al. (2001). Therefore, it would be extremely challenging to model such complex intraday return dynamics as their random characters may be further complicated by market microstructure noise. Addressing this challenge, we propose a novel semiparametric approach, called the functional autoregressive value-at-risk (hereafter, FARVaR) for forecasting the daily return density and VaR using the high frequency intraday information explicitly. We first construct a nonparametric intraday density and apply the functional autoregressive (hereafter, FAR) model² to forecast the intraday density. We then obtain the daily return density forecast directly out of the intraday density forecast using either the parametric approximation based on the general class of the Normal Inverse Gaussian distribution³ or the nonparametric bootstrap, from which we measure the daily VaR.

FARVaR has three main advantages as follows. First, it avoids any (distributional) uncertainty associ-

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¹For example, Brogaard et al. (2015) shows that HFT facilitates price efficiency by trading in the direction of permanent price changes and in the opposite direction of transitory pricing errors. It suggesting that HFT is beneficial to price discovery and market quality, a similar conclusion drawn by Hasbrouck and Saarb (2013). In contrast, Zhang (2010) shows that HFT is positively correlated with stock price volatility and this positive correlation is even stronger during periods of high market uncertainty. He therefore argues that HFT is harmful for price discovery.

²Bosq (2000) introduces the statistical foundation of the FAR modelling. See also Park and Qian (2007).

³The NIG distribution is one of the most popular distribution family for describing the return distribution in the literature (Bandorff-Nielsen, 1997).
ated with misspecified parametric models by estimating the intraday density nonparametrically. Furthermore, FAR can easily overcome shortcomings of the parametric models by capturing such complex dynamic structure via a functional autoregressive operator, which can represent all possible contemporaneous and time-dependent associations among all the moments or quantiles (Park and Qian, 2007).

Second, the high frequency financial data is often characterized as extremely dispersed and non-normally distributed (Hasbrouck, 2007). Based on a detailed exploratory analysis of the time-varying intraday moments we find two stylized facts: (i) volatility and skewness of intraday returns are rather persistent; (ii) there exist complex (potentially nonlinear) and time-varying associations among the moments. In this regard, either parametric or nonparametric approach is inappropriate to unravel an exact relationship among intraday moments. By contrast, FARVaR is designed to utilize intraday price evolution directly into forecasting a daily VaR in a flexible manner: the stylized facts of intraday returns thoroughly flow into the FAR modelling of the intraday return density and then the intraday density forecast by FAR is utilized for constructing the daily return density and measuring the daily VaR. Therefore, FARVaR can offer an ideal framework to abstract the daily density and VaR from intraday returns in a robust manner.

Third, it is well established that parametric models can reduce the economic cost but tend to underestimate VaR (Netftci, 2000) while nonparametric models can provide conservative coverage ability but fail to reduce the economic cost. This fundamental trade-off may reflect their respective forecasting inaccuracy. As confirmed by our empirical application, we find that the semiparametric FARVaR approach can improve the coverage ability and reduce the economic cost simultaneously.

We conduct various evaluation schemes using both simulated and real data of 30 stocks listed in Dow Johns Industry Average (DJIA) over a period, 2000–2008. All the evaluations are based on the out-of-sample forecasting. First, we evaluate the intraday density forecasting performance of alternative functional models by comparing divergence criteria. We find that FAR is the best predictor of the intraday density as it produces the smallest divergence among the functional models including a functional martingale process (hereafter, LAST) and a functional i.i.d. process (hereafter, AVE).

Second, we conduct the Monte Carlo experiments using a simulated sample constructed on the basis of the empirical distribution, and evaluate the VaR forecasts against the true VaR. In terms of root mean square error, mean absolute error, and mean absolute percentage error, we document that FARVaR produces the smaller forecasting errors than the existing VaR models.

Furthermore, we apply a wide range of backtesting tools to evaluate the performance of FARVaR against

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4Recently, Hallam and Olmo (2014) propose an alternative approach for forecasting the daily return density (directly generated from transformed intraday returns) by a simple autoregressive model on the daily return density, which can be seen as a special case of our proposed FAR model.
the existing VaR models. A backtest is not only a formal framework for verifying whether the actual loss is in line with the projected loss, but also an essential procedure for selecting suitable internal VaR models for capital requirements as recommended by the Basel committee. To this end, we conduct three board type of tests. We examine the coverage ability and the economic cost using the conventional quantitative measures such as the empirical coverage probability (ECP) and the predictive quantile loss (PQL; Koenker and Bassett, 1978). Furthermore, we assess the Basel penalty zone (BPZ) and the market risk capital requirement (MRCR; BCBS, 1996, 2004). In addition, we evaluate the statistical adequacy of the VaR models with the conditional coverage test (CC test; Christoffersen, 1998) and the dynamic quantile tests (DQ test; Engle and Manganelli, 2004). These backtesting results demonstrate that FARVaR is the most robust VaR model. It strengthens the coverage ability, reduces the economic cost, and improves the statistical reliability, simultaneously.

2 FARVaR

The aim of the study is to develop a novel risk management model for forecasting the daily VaR using intraday information explicitly. In this context, an important but nontrivial issue is how best to develop the appropriate econometric methodology for estimating and forecasting the daily return density using intraday returns. The most popular approach is to construct realized volatility from intraday returns and forecast the daily volatility by applying a parametric model to the realized volatilities. However, this approach uses only realized volatilities such that it is not sufficiently general for generating the entire density. Importantly, this approach tends to suffer from potentially misspecified parametric assumptions imposed on the dynamics of realized volatility and the distribution of daily returns.

More recently, Hallam and Olmo (2014) propose an alternative approach for estimating the daily return density directly from intraday returns by rescaling them through multiplying the scaling factor under the assumption that the intraday return process is self-affine or unifractal. They estimate the daily density through applying either a location-scale t-distribution or the kernel density estimator to the transformed daily returns. They then forecast the daily density by estimating distributional parameters via a simple autoregressive model or as the weighted sum of past daily densities. Although their approach is more general those in the previous studies, it is more restrictive than our proposed approach in two ways. First, the transformed (daily) returns should be equal to the cumulative sum of intraday returns by construction, though the higher order moments such as skewness and kurtosis of both transformed returns and intraday

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5For example, Giot and Laurent (2004) use an ARCH type model, Clements et al. (2008) consider the mixed data sampling and heterogeneous autoregressive model, and Andersen et al. (2003) employ a bivariate VAR.
returns are equivalent under a common rescaling. This would violate the market microstructure’s stylised evidence that intraday returns are more skewed and leptokurtic than daily returns. Second, their proposed daily density forecasting approach is clearly more restrictive than the functional autoregressive (FAR) modelling.\textsuperscript{6}

In the following we develop the semiparametric FARVaR methodology, which generates the daily density forecast directly from the intraday density forecast. It consists of the two steps which is illustrated in Figure 1. First, we estimate the density of intraday returns nonparametrically via the kernel density estimator and forecast the intraday density by FAR. Second, we construct the daily return density directly from the intraday density forecast and measure the daily VaR. To this end we propose two approaches - a parametric approximation based on the general class of the Normal Inverse Gaussian distribution and a nonparametric bootstrap - both of which can transform intraday density function into the daily counterpart in a flexible manner.

2.1 Forecast Intraday Density Function by FAR

Bosq (2000) introduces the statistical foundation of the FAR modelling. Its asymptotic theory is refined by Cardot et al. (2007) and Mas (2007). In the early stage, FAR is applied for forecasting the climate pattern such as temperature (Besse et al., 2000) and ozone (Aneiros-Perez et al., 2004; Damon and Guillas, 2002). Recently, the FAR approach is adopted in economics and finance. Laukaitis (2008) applies it to forecasting the intraday cash flow and the intensity of the transaction in a credit card payment system. Bowsher and Meeks (2008) and Kargin and Onatski (2008) employ it to forecast a yield-curve as a function of maturity. Chaudhuri et al. (2015) apply FAR to the cross-sectional density functions of sectoral inflation rates estimated nonparametrically, and develop the flexible framework for forecasting the density of national inflation rates.

We now describe a detailed procedure for estimating and forecasting intraday return density. Using a series of asset prices observed at a fixed time interval (e.g., 5 minute), we calculate an intraday return by differencing the logged price,\textsuperscript{7}

\begin{equation}
    r_{ti} = \ln P_{ti} - \ln P_{t,i-1}, \quad t = 1, \ldots, T; \quad i = 1, \ldots, m
\end{equation}

\textsuperscript{6}Chaudhuri et al. (2015) show that the simple average of past density functions are inferior to the FAR in forecasting the density of inflation rates.

\textsuperscript{7}In practice, the trading action and its frequency are irregular in time, so we filter the data and draw prices in a fixed time base, e.g., 9:35AM, 9:40AM, ..., 4:00PM.
where \( t \) and \( i \) denote the \( t \)th day and its \( i \)th intraday observation, respectively. Since \( P_{t,0} = P_{t-1,m} \) denotes the closing price of the previous day, overnight information is incorporated in the first observation \((r_{t1})\) at the opening time. Further, a daily return is calculated as the sum of the intraday returns:

\[
r_t = \ln P_{tm} - \ln P_{t-1,m} = \sum_{i=1}^{m} r_{ti}, \quad t = 1, \ldots, T. \tag{2}
\]

In consequence, we have the following sequence of time series:

\[
X = (X_t)_{t=1}^{T} \quad \text{and} \quad X_t = (r_{ti})_{i=1}^{m}, \tag{3}
\]

where \( X \) is a set of intraday return paths and \( X_t \) is the intraday return path at the \( t \)th day. We assume that the intraday return is stationary within a given day such that the intraday density can be consistently estimated. Denoting the intraday density function by \( f_t \), the sequence, \((f_t)_{t=1}^{T}\), can be defined in a functional space. But, we do not allow this local stationarity to carry over a longer horizon, i.e., the intraday density is varying over time. This is known as the “piecewise stationarity” of a stochastic process (Gabor, 1946; Park and Qian, 2007). In general, a “piece” is taken as any time unit such as a day, week or month.

We now model the time-varying intraday density function \((f_t)\) by the FAR process of order one:

\[
w_t = Aw_{t-1} + \epsilon_t, \quad t = 1, \ldots, T, \tag{4}
\]

where \( w_t (= f_t - \mathbb{E}[f]) \) is a fluctuation of the density function from the well-defined common expectation of the density function, \( A \) is an autoregressive operator in the Hilbert space \((\mathcal{H})\), satisfying \( \|A^\kappa\| < 1 \) for \( \kappa > 0 \) and \((\epsilon_t)_{t=1}^{T}\) is a sequence of the functional white noise process. Then, (4) can be rewritten in terms of the density function:

\[
f_t = \mathbb{E}[f] + Aw_{t-1} + \epsilon_t. \tag{5}
\]

The one-step ahead forecast of the density function can be evaluated by the conditional expectation on the past information set \((F_{t-1})\):

\[
\mathbb{E}[f_t|F_{t-1}] = \mathbb{E}[f] + Aw_{t-1}. \tag{6}
\]

If \( A \) is a zero operator, the best predictor would be the unconditional expectation of the density (i.e. \( AVE \)). On the other hand, if \( A \) is an identity operator, the last observation would be the best predictor (i.e. \( LAST \)).

\[
AVE : \mathbb{E}[f_t|F_{t-1}] = \mathbb{E}[f], \quad \text{LAST : } \mathbb{E}[f_t|F_{t-1}] = f_{t-1}. \tag{7}
\]
FAR can be regarded as a general autoregressive modelling for (higher-order) moments or quantiles.

To illustrate, we represent the centered mean in (4), \( \bar{\mu}_t = \langle x, w_t \rangle \), by

\[
\bar{\mu}_t = \langle x, Aw_{t-1} \rangle + \langle x, \epsilon_t \rangle = \langle A^* x, w_{t-1} \rangle + \eta_t,
\]

where \( A^* \) is the adjoint of \( A \), \( \langle \cdot, \cdot \rangle \) is the inner product on \( H \) such that \( \langle \nu, u \rangle = \int_R \nu(x) u(x) \, dx \) and \( (\eta_t)_{t=1}^T \) is a white noise process. Then, \( \langle A^* x, w_{t-1} \rangle \) can be represented by an infinite sum of polynomials,

\[
\sum_{k=1}^{\infty} c_k \langle x^k, w_{t-1} \rangle = \sum_{k=1}^{\infty} c_k \bar{\mu}_{k,t-1},
\]

where \( \bar{\mu}_{k,t-1} \) is the first-lagged centered \( k \)th order moment. This representation implies that the centered mean is specified by a linear combination of all of first-lagged higher-order moments. Similarly, the centered second order moment, \( \bar{\mu}_{2t} = \langle x^2, w_t \rangle \), can be expressed by a linear combination of first-lagged higher order moments. Consequently, the autoregressive models for higher-order moments can be regarded as a special case of FAR. See Park and Qian (2007) who derive ARCH and ARCH-M as the special case of FAR. Notice that the quantile process, \( (q_t)_{t=1}^T \), also tends to revert to the \( \alpha \)th quantile of \( E[f] \) as long as \( f_t \) reverts to \( E[f] \) in (5), where \( q_t \) satisfies \( \alpha = \int_{-\infty}^{q_t} f_t(x) \, dx \). Hence, FAR can represent a very general autoregressive dynamic structure, suitable for modelling complex time-varying return distributions in a robust manner.

To apply FAR, we first estimate an intraday density at each point of time \( t \) via the nonparametric kernel density estimator:

\[
\hat{f}_t(x_j) = \frac{1}{nh_t} \sum_{i=1}^{m} K \left( \frac{x_j - r_{ti}}{h_t} \right), \quad t = 1, \ldots, T; \quad j = 1, \ldots, n,
\]

where \( K \) is a kernel, \( m \) is the number of observations, \( h_t \) is a bandwidth (smoothing parameter or window width) and \( n \) is the number of discrete grids.\(^8\) One practically important issue is the selection of kernel and corresponding bandwidth. An optimal bandwidth is derived by minimizing the loss function and applying a cross validation selector. We employ the popular Gaussian kernel and follow the Silverman’s (1986) rule of thumb in which case the optimal bandwidth is given by \( 1.06\hat{s}_t m^{-1/5} \) with \( \hat{s}_t \) being the sample standard deviation of \( r_{ti} \). Given the sequence of the estimated density functions, \( \left( \hat{f}_t \right)_{t=1}^T \), we estimate \( E[f] \) by the sample average of \( \hat{f}_t \), i.e. \( \bar{f} = T^{-1} \sum_{t=1}^{T} \hat{f}_t \). The fluctuation is then obtained by \( \hat{w}_t = \hat{f}_t - \bar{f} \).

\(^8\)The grid set covers the range of sample, \( x_1 < x_2 < \ldots < x_n \). For simplicity, we use an equal interval, \( \delta = x_j - x_{j-1} \) for all \( j = 1, \ldots, n \).
Next, the autoregressive operator, $A$, in (4) can be obtained theoretically by $A = C_0^{-1}C_1$, where $C_0$ and $C_1$ are autocovariance operators of order 0 and 1, respectively (Bosq, 2000; Park and Qian, 2007). Since the autocovariance operators are defined in the infinite dimension, it is impossible to compute $C_0^{-1}$ in practice. To avoid this ill-posed inverse problem, we project $C_0$ into a finite $L$-dimensional subspace by a functional principal component analysis, and then obtain the consistent estimator of $A$ in the $L$-dimensional subspace, denoted by $\hat{A}_L$ (see Bosq, 2000; Park and Qian, 2007). Then, the one-step ahead conditional density forecast is evaluated by

$$\hat{f}_{T+1} = \bar{f} + \hat{A}_L \hat{w}_T,$$

where $\hat{f}_{T+1}$ denotes $\mathbb{E}\left[f_{T+1} | F_T\right]$ for the simplicity.

We often approximate $\hat{w}_t$ by the Fast Fourier Transformation (hereafter, FARFFT) or the Wavelet transformation (hereafter, FARWV). These approximations can reduce the computing time substantially through shrinking the dimension of the function (about thirty times faster). After obtaining the forecast of the transformed $\hat{w}_{T+1}$, we invert it to the original one (Antoniadisa and Sapatinas, 2003; Besse et al., 2000).

### 2.2 Construct Daily Density Function from Intraday Density Forecast

If the parametric distribution of returns were known, it would be straightforward to generate the daily density from intraday density. In this context we propose the parametric approximation on the basis of the Normal Inverse Gaussian (NIG) distribution, regarded as one of the most flexible distribution family for describing the return distribution (Bandorff-Nielsen, 1997). In the general case where the analytic form of the parametric distribution is unknown, we propose the nonparametrical bootstrap. Asymptotically, the latter approach is expected to be more efficient than the former because the simulation approach attempts to utilize all the information from intraday density and the NIG approximation involves a potentially strong distributional assumption.

#### 2.2.1 The NIG Approximation Approach (FARVaR-NIG)

This approach attempts to construct the daily return density from the first four moments of a intraday density, using the parametric approximation on the basis of NIG. Let $\mu_t$, $\nu_t$, $s_t$ and $k_t$ be, respectively, the mean, variance, skewness, and kurtosis. Then, we calculate the four parameters, $(\alpha_t, \beta_t, \gamma_t, \delta_t)$, that
determine the shape of the NIG distribution using the following formula:

\[
\alpha_t = v_t^{-\frac{1}{2}} (3k_t - 4s_t^2 - 9)^{-\frac{1}{2}} \left( k_t - \frac{5}{3}s_t^2 - 3 \right)^{-1}, \quad \beta_t = s_t v_t^{-\frac{1}{2}} \left( k_t - \frac{5}{3}s_t^2 - 3 \right)^{-1},
\]

\[
\gamma_t = \mu_t - 3s_t v_t^2 \left( 3k_t - 4s_t^2 - 9 \right)^{-1}, \quad \delta_t = 3^{\frac{3}{2}} \left\{ v_t \left( k_t - \frac{5}{3}s_t^2 - 3 \right) \right\}^{\frac{1}{2}} \left( 3k_t - 4s_t^2 - 9 \right)^{-1}, \quad (12)
\]

where \( \alpha_t \) determines the tail heaviness, \( \beta_t \) the asymmetry, \( \gamma_t \) the location, \( \delta_t \) the scale of the distribution. Notice that the kurtosis should satisfy: \( k_t > 3 + (5/3) s_t^2 \). The intraday density function is then approximated by the following NIG density:

\[
f_t(x) = \left[ \frac{\alpha_t \delta_t J_1 \left( \alpha_t \sqrt{\delta_t^2 + (x - \gamma_t)^2} \right)}{\pi \sqrt{\delta_t^2 + (x - \gamma_t)^2}} \right] e^{\delta_t \lambda_t + \beta_t (x - \gamma_t)}, \quad (13)
\]

where \( \lambda_t = \sqrt{\alpha_t^2 - \beta_t^2} \) and \( J_1 \) denotes the modified Bessel function of the second kind.

The NIG distribution is close under the convolution of independent random variables \( X \) and \( Y \):

\[
X \sim NIG(\alpha, \beta, \gamma_X, \delta_X), \quad Y \sim NIG(\alpha, \beta, \gamma_Y, \delta_Y) \Rightarrow X + Y \sim NIG(\alpha, \beta, \gamma_X + \gamma_Y, \delta_X + \delta_Y). \quad (14)
\]

Hence, the density function of the daily return, \( r_t (\sum_{i=1}^m r_{it}) \), can be approximated by the following NIG density:

\[
g_t(x) = \left[ \frac{m \alpha_t \delta_t J_1 \left( m \alpha_t \sqrt{\delta_t^2 + (x - m \gamma_t)^2} \right)}{\pi \sqrt{m^2 \delta_t^2 + (x - m \gamma_t)^2}} \right] e^{m \delta_t \lambda_t + \beta_t (x - m \gamma_t)}, \quad (15)
\]

where \( m \) is the number of intraday return observations within a day.

Finally, we can obtain the daily VaR from the cumulative NIG density function using the four parameters, \( \left( \hat{\alpha}_{T+1}, \hat{\beta}_{T+1}, \hat{\gamma}_{T+1}, \hat{\delta}_{T+1} \right) \), obtained from the intraday density forecast.

2.2.2 The Simulation Approach (FARVaR-SIM)

We simulate intraday returns from the intraday density and calculate the daily return by the cumulated sum of intraday return draw. Repeating this random sampling, we can construct the daily density and the daily VaR. We first construct the intraday cumulative density function from the intraday density forecast as

\[
\hat{F}_{T+1}(x) = \int_{-\infty}^{x} \hat{f}_{T+1}(x) \, dx. \quad (16)
\]
Since \( \hat{f}_{T+1} \) is the discrete approximation of the continuous density function, we approximate \( \hat{F}_{T+1} \) numerically by the middle Riemann sum:

\[
\hat{F}_{T+1}(z_j) = \frac{1}{2} \left[ \sum_{i=1}^{j} \hat{f}_{T+1}(x_{i+1}) \Delta x_{i+1} + \sum_{i=1}^{j} \hat{f}_{T+1}(x_i) \Delta x_{i+1} \right], \quad j = 1, \ldots, n - 1,
\]

(17)

where \( z_j = (x_{j+1} + x_j) / 2 \) and \( \Delta x_{i+1} = x_{i+1} - x_i \). Given the piece-wise stationarity of intraday returns, we simulate intraday returns from \( \hat{F}_{T+1} \) as follows: First, generate \( m \) numbers from the uniform distribution over \((0, 1)\), denoted by \( \{p_{1}^{(b)}, p_{2}^{(b)}, \ldots, p_{m}^{(b)}\} \), where \( p_{i}^{(b)} \sim U(0, 1) \) for each trial \( b = 1, \ldots, B \). Next, we let \( j^* \) be an index \( j \), satisfying

\[
j^* = \arg\min_{j \in \{1, \ldots, n-1\}} \left| \hat{F}_{T+1}(z_j) - p_{i}^{(b)} \right|.
\]

(18)

We then draw a grid \( \hat{z}_{j^*}^{(b)} \) corresponding to the index \( j^* \) from the set \( \{z_1, z_2, \ldots, z_{n-1}\} \) such that we construct a set of \( m \) simulated intraday returns \( \{\hat{z}_{1}^{(b)}, \hat{z}_{2}^{(b)}, \ldots, \hat{z}_{m}^{(b)}\} \). Then, we construct the \( b \)th daily return by \( r^{(b)} = \sum_{j=1}^{m} \hat{z}_{j}^{(b)} \). Repeating this procedure \( B \) times, we can approximate the (empirical) cumulative density function of the daily return by

\[
\hat{G}_{T+1}(\omega) = \frac{1}{B} \sum_{b=1}^{B} 1 \left\{ r^{(b)} \leq \omega \right\}.
\]

(19)

Finally, we are able to evaluate the daily VaR forecast by

\[
\hat{VaR}_{T+1}(\alpha) = \inf \left( \omega | \hat{G}_{T+1}(\omega) \geq 1 - \alpha \right),
\]

(20)

where \( \alpha \) is a given nominal probability, e.g. 1% or 5%.

### 3 The Data

In our empirical application, we consider the constituents of the Dow Jones Industrial Average (DJIA) index over the period of 2000 - 2008.\(^9\) These corporations are deemed to be too big to fail even during the financial crisis. Three companies were rescued by the U.S. government in 2008: Bailouts for American International Group Inc. and Citi group Inc. were announced by the Federal Reserve Bank’s Board

\(^9\)Note that the data for Hewlett-Packard Company is collected from 06/May/2002 to 31/Dec/2008 and the data for Version-Communication Inc. is collected from 03/Jul/2000 to 31/Dec/2008.
of Governors. Though the case of General Motors Corporation is not technically a bailout, a bridge loan was given by the U.S. government in 2008, which is technically referred to as a bailout. It is important to understand the risk profiles of these companies so as to monitor and safeguard the US financial system. Furthermore, these companies are actively traded so that their transactions are likely to generate enormous amount of information even at an intraday level. Hence, we aim to address an important empirical question of whether and how the use of intraday information can improve the risk management modelling in practice.

The constituents of the DJIA change annually. We choose the benchmark list of constituents in 2005, which corresponds to the middle of the whole sample period. Intraday transactions data are collected from Trade and Quote (TAQ) database. Following existing studies, e.g., Lee and Ready (1991) and Hvidkjaer (2006), we have also applied a filtering procedure in order to exclude any data likely to be erroneous. Specifically, all the trades (quotes) with condition codes, A, C, D, G, L, N, O, R, X, Z, 8, 9 (4, 5, 7–9, 11, 13–17, 19, 20) are eliminated. The trades with a correction code which is greater than 2 are also removed (refer to the TAQ manual for the definition of the codes). Quotes are excluded if ask is equal to or less than bid, or bid, if ask spread is above 75% of mid-quote, or if ask (bid) is more than double or less than half of the previous ask (bid). We only consider trades reported from 9:30 AM to 4:00 PM. According to Lee and Ready (1991), trades occur five seconds earlier than reported time. Thus, we calculate trade time as the reported time minus 5 seconds. Trades are also deleted if the trade price is more than double or less than half of the previous trade. After filtering data, close trade and quote prices in every 5 minute intervals have been calculated. Close trade and quote prices are defined as the price of the last trade in intervals and corresponding quote price when trade occurs. If there is no trade, the price for the current interval is replaced by the closest trade and quote prices of the previous interval.

3.1 Descriptive Statistics

Table 1 presents the descriptive statistics for intraday and daily returns of 30 stocks over the period of 2000 - 2008. Panel A reports the results for intraday returns. The mean returns are very close to zero, a consistent finding in the market microstructure literature. Standard deviation of Intel Corporations is also the highest. There is significant evidence of asymmetry as skewness different from zero and mainly negative. As expected, kurtosis is significantly much larger than that of the normal distribution, highlighting the typical fat-tails of the financial data.

Panel B reports the descriptive statistics for the daily return. The mean daily return is small and generally around a few basis points. Average returns of American International Group (-0.17%) and Citigroup...
(-0.08%) are relatively lower. This is not surprising as they were rescued by Fed in 2008. The maximum (minimum) and the standard deviation of these companies are also much larger (smaller). The asymmetry of the return distribution is not as severe as those in the intraday, though AIG (-6.49) and Procter & Gamble Company (-5.17) are substantially negatively skewed. Fat-tails are also observed for all the companies. In particular, the kurtosis of AIG (158.5), Citigroup (47.82) and Procter & Gamble (120.3) are significantly higher. Overall, these findings not only confirm the typical patterns of financial time series, but also display some extreme statistics especially for companies bailed out during the global financial crisis in 2008.

3.2 Analysis of Time Varying Intraday Moments

The existence of a trade-off between risk and expected return is central to modern finance, though there is little empirical consensus on whether the risk-return trade-off is negative (e.g. Glosten et al., 1993) or positive (Ludvigson and Ng, 2007). Furthermore, skewness may play an important role in explaining the variation of excess returns (e.g. Harvey and Siddique, 2000), but most existing studies tend to ignore the issue of whether and how skewness will affect the relation between expected return and risk.

To further enhance our understanding of complex intraday return dynamics, we now provide the time-varying descriptive analysis of the four centre moments (mean, standard deviation, skewness and kurtosis) of intraday returns of the equal weighted portfolio of 30 asset returns.

This analysis is expected to provide the stylised features amongst all the moments and their time-varying associations in a flexible manner.

In Figure 2 we display the autocorrelation function (ACF) of the volatility and the skewness, respectively. As expected, volatility measured by intraday variance exhibits high persistence with its AR(1) coefficient around 0.76. This is a consistent finding with those documented in Andersen et al. (2001). The skewness is weakly persistent as its AR(1) coefficient is slightly negative at -0.074, but statistically significant. On the other hand, the ACFs of mean and kurtosis are statistically insignificant (not reported to save space).

We now turn to analyzing the time-varying patterns of contemporaneous correlation among the moments. Figure 3 displays scatter plots between the pair of the moments. First, from Panel (a), we find an U-shaped relation between return and volatility, suggesting that the correlation is regime-dependent.

10We have also conducted the same exercises using the value weighted portfolio, finding qualitatively similar results.
and measured at 0.672 when the mean is positive (the bull market) and -0.633 when mean is negative (the bear market).\textsuperscript{11} This finding cast doubt on the strongly held intuition that the risk-return trade-off should be positive at the market-wide level (e.g. Campbell, 1987), according to which higher risk should be accompanied by higher return. The U-shaped risk-return relation observed at the intraday frequency is also in line with the market microstructure model (e.g. Hasbrouck, 2007, Chapter 2) where the unconditional mean revert to zero at a high frequency step. It suggests that high-frequency trading strategy should be a directional bet with information advantage (e.g. knowledge about order flows) since bearing additional volatility risk will not be necessarily compensated by higher returns. This evidence suggests that the imposition of the time-invariant and linear risk-return trade-off would lead to inaccurate and misleading forecasts.

Second, Panel (b) shows that mean and skewness are positively associated with the correlation coefficient of 0.512. There are mixed evidences on the return-skewness relationship mostly at the lower frequency (weekly or monthly) level, e.g. Conrad et al. (2013) and Rehmany and Vilkovz (2012). To the best of our knowledge, however, there is no documented evidence at the intraday level. Intuitively, our finding suggests that extreme values or outliers are likely to play a more significant role in generating intraday returns such that a few large positive (or negative) movement will render the average return moving in the same direction.

Third, we find from Panel (c) that there also exists an U-shaped relation between volatility and skewness with the correlation measured at -0.366 when skewness is negative (downside risk) and at 0.227 when skewness is positive (upside uncertainty). Given that we obtain the U-shape risk-return trade-off and the linear return-skewness trade-off, this finding is consistent by transitivity. It further confirms that extreme large price movements (positive or negative skewness) are associated with higher volatility in the market. The steeper slope observed under downside risk also suggests that the volatility-skewness relationship is stronger when the market is hit by bad news.

Fourth, Panel (d) displays a strong quadratic U-shape relation between skewness and kurtosis. The correlation is measured at -0.963 when skewness is negative and at 0.977 when skewness is positive. From the probability distribution theory, we expect the following quadratic boundary condition to hold: Assuming that the mean is zero and the standard deviation is unity, the maximum boundary condition is defined as kurtosis being the quadratic function of skewness, namely, \( k > s^2 + 1 \) (Jondeau and Rockinger, 2003). In the more general case with the NIG distribution, the boundary condition is modified as \( k > (5/3) s^2 \).

\textsuperscript{11}This finding is qualitatively consistent with the recent literature documenting a non-monotonic relation between return and volatility, e.g. Rossi and Timmermann (2012) and Chiang and Li (2012).
(Bandorff-Nielsen, 1997). Hence, their strong quadratic U-shape relation is mathematically a pleasing finding.

[FIGURE 3 ABOUT HERE]

Overall, our time-varying moments-based analysis reveals two stylized facts. First, we find that the second and third moments of intraday return density are persistent. Second, contemporaneous associations among the four moments are all complex and time-varying. These results provide a strong support for the numerous lower-frequency studies showing that the time-varying characteristics of the higher order moments should be carefully modelled for portfolio allocation and the asset pricing purpose (Cenesizoglu and Timmermann, 2012; Harvey and Siddique, 2000). This also highlights the challenge of modelling intraday return dynamics. A nonparametric approach tends to ignore the time-varying nature of the density function, which leads to an inaccurate forecasting. The fully parametric approach, suffering from the misspecification, is inappropriate to unravel an exact relationship among the higher-order moments, which results in the erroneous measurement of risk. In this regard, our proposed FARVaR approach is expected to take into account the relative advantages of both approaches and thus reflect such complex characteristics of intraday returns in modelling and forecasting the intraday density in a robust manner.

4 The Pseudo Out-of-Sample Forecasting Evaluations

In this section we conduct the pseudo out-of-sample forecasting exercises to evaluate the forecasting performance of the number of functional models proposed above (namely, FAR, FARFFT, FARWV, AVE and LAST). This practice of holding out sample is called “pseudo real time” experiments (e.g. Elliot and Timmermann, 2008). To this end we first evaluate the intraday density forecast performance of the functional models via comparing divergence criteria. Next, we evaluate the VaR forecast performance with respect to the simulated data on the basis of the forecasting errors between the VaR forecast and the true one.

4.1 Intraday Density Forecast Performance

To examine which of the five functional models described above can produce the most accurate forecasts of intraday density, we evaluate the three most popular divergence criteria; the Hilbert norm \(D_H\), the uniform norm \(D_U\), and the generalized entropy \(D_E\) which measure the distance between the forecasted
and the true density function:

\[
D_H (\hat{f}, f) = \frac{\int (\hat{f}(x) - f(x))^2 \, dx}{\int \hat{f}(x)^2 \, dx + \int f(x)^2 \, dx}, \quad D_U (\hat{f}, f) = \frac{\sup_x |\hat{f}(x) - f(x)|}{\sup_x f(x)},
\]

\[
D_E (\hat{f}, f) = \int f(x) g \left( \frac{\hat{f}(x)}{f(x)} \right) \, dx,
\]

(21)

where \( \hat{f}(f) \) denotes the forecasted (realized) density function and \( g(y) = (\gamma - 1)^{-1} (y^\gamma - 1) \) with \( \gamma > 0 \) and \( \gamma \neq 1 \). We follow Park and Qian (2007) and set \( \gamma = 1/2 \). If \( g \) is the natural log, this becomes the Kullback-Liebler divergence measure. All three quantities are non-negative (called the global error) and produce a zero value if \( \hat{f} = f \). \( D_H \) is useful for evaluating the goodness-of-fit of the model, \( D_U \) is informative for comparing the closeness of the function shape and \( D_E \) assesses the difference in information content.

Here and throughout the paper, we apply the rolling forecasting approach with the window size of 250 to accommodate any time-varying or regime-switching patterns. To begin with we estimate the five functional models over the period 3 Jan. 2000 - 27 Dec. 2000 and compute one-day-ahead intraday density forecast for 28 Dec. 2000. We then repeat this procedure by moving forward a day at a time in a rolling manner and ending with the density forecast for 31 Dec. 2008. This generates 2,013 daily forecasts except for Hewlett-Packard, AT&T and Verizon Communication with 1,428, 2,006 and 1,887 density forecasts, respectively.

Table 2 presents the evaluation results in terms of both mean and median forecasts for 30 stocks. Overall, we find that the FAR models outperform other functional models, LAST and AVE. In particular, FARFFT turns out to be the best predictor as it produces the smallest divergence. Therefore, we will mainly consider the FARFFT approach in the further analysis of VaR.

4.2 VaR Forecast

We now conduct the simulation study to evaluate the forecasting performance of FARVaR against the number of existing VaR models which are popularly employed by both academics and practitioners: Historic simulation (HS), Filtered historic simulation (FHS), RiskMetrics (RiskMetrics, 1996; hereafter, RM), GARCH, the Filtered Extreme Value Theory (hereafter, FEVT) models, conditional autoregressive value at risk by regression quantiles (CAViaR) and CAViaR-GARCH. HS is a static nonparametric model and most
popular for simplicity (Perignon et al., 2008). RM and GARCH are typical dynamic parametric models. RM assumes the normal distribution of asset returns while GARCH can also allow the fat-tailed Student’s t distribution. FHS is the hybrid approach consisting of two steps by applying HS to returns filtered by GARCH. FEVT is suggested to control for time-varying volatility (Diebold et al., 1998; McNeil, 2000). Similarly to FHS, it applies the EVT procedure to returns filtered by GARCH. In our analysis we consider the filtered generalized extreme value (hereafter, FGEV) distribution and the filtered generalized Pareto distribution (hereafter, FGPD). Finally, we examine CAViaR that uses the symmetric specification of returns and CAViAR-GARCH that embodies GARCH in CAViaR (see Engle and Manganelli, 2004). In the literature FHS has been found to be one of the most successful VaR models (e.g. Barone-Adesi et al., 2002; Kuester et al., 2006; Pritsker, 2006).

We evaluate the RMSE, MAE and MAPE of the forecasting errors generated by alternative models:

\[
RMSE = \sqrt{\frac{\sum_{t=1}^{N} d_t^2(\alpha)}{N}}, \quad MAE = \frac{\sum_{t=1}^{N} |d_t(\alpha)|}{N}, \quad MAPE = \frac{\sum_{t=1}^{N} \left| \frac{VaR_t(\alpha)}{VaR_t(\alpha)} - 1 \right|}{N},
\]

where \(\alpha\) is the selected quantile (say 1 or 5%), \(d_t(\alpha) = \hat{VaR}_t(\alpha) - VaR_t(\alpha)\), \(\hat{VaR}_t(\alpha)\) denotes the VaR forecast based on the information up to \(t - 1\), and \(VaR_t(\alpha)\) denotes the true VaR.

The construction of the appropriate data generating process (DGP) is nontrivial. Hence, we make the following simplifying assumptions:

**Assumption 1.** Intraday returns are independently and identically distributed.

**Assumption 2.** The distribution of intraday return is characterized mainly by the first four moments such that it follows the NIG distribution.

**Assumption 3.** The first four moments follow a stationary vector autoregressive process of order one.

**Assumption 4.** The daily return is the sum of the intraday returns and thus follows a NIG distribution.

First, we generate the DGP of the first four moments of the intraday return to follow the VAR(1) process:

\[
(m_t - \mu) = \Phi (m_{t-1} - \mu) + \xi_t, \quad t = 1, \ldots, T,
\]

where \(m_t = (\mu_t, v_t, s_t, k_t)^t, \quad m = (\mu, v, s, k)^t\) is the unconditional mean vector of moments and \(\Phi\) is an autoregressive coefficient matrix satisfying \(\|\Phi^k\| < 1\) for any \(k \geq 1\). \(\xi_t = (\xi_{t1}, \xi_{t2}, \xi_{t3}, \xi_{t4})^t\) is independently and normally distributed with zero mean and covariance, \(\Sigma = diag(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2)\).\(^{12}\)

\(^{12}\)We impose zero covariance on the off-diagonal of covariance matrix \(\Sigma\) for simplicity but we also find qualitatively consistent simulation results from non-zero covariance.
Second, we generate the vector of sample moments by
\[
m_t = (I - \Phi)m + \Phi m_{t-1} + \xi_t,
\]
(24)
where the initial vector \(m_0\) is simply given by the zero vector. Notice that the moments must satisfy the following conditions, \(v_t > 0\) and \(k_t > \frac{5}{2} s_t^2\) (see the definition of NIG distribution). We discard the first 100 observations to burn out the effect of the initial values.

Next, we calculate the four parameters, \((\alpha_t, \beta_t, \gamma_t, \delta_t)\)' that determine the NIG distribution, using the four moments, \((\mu_t, v_t, s_t, k_t)\)' (see (12)). Then, we randomly draw intraday returns from the NIG distribution and construct \(X_t = (r_t)_{i=1}^n\) and \(X = (X_t)_{t=1}^{T+N}\) (see (3) for the definitions of \(X_t\) and \(X\)). Under Assumptions 1 - 4, the NIG distribution of daily returns is determined by four parameters \((\alpha_t, \beta_t, m\gamma_t, m\delta_t)\)'. Hence, we can calculate a true daily \(VaR\) at \((1 - \alpha)\)% from an inverse cumulative density function given the nominal probability \(\alpha\).

Next, we evaluate the one-step-ahead \(VaR\) forecast of \(FARVaR\) at the 99% level against the forecasts produced by the alternative \(VaR\) models using the rolling window size of 250. With 5,000 iterations, the relative forecasting performances are compared through accessing the forecasting errors measured as the difference between the \(VaR\) forecast and the true \(VaR\).

In principle, the parameter values of \(m, \Phi\) and \(\Sigma\) are unknown and difficult to identify a priori. Hence, to select these parameter values, we follow the data-oriented approach and estimate (24) for each of 30 stocks as well as their equal weighted portfolio. We decide to focus on three different cases. First, we consider the estimation results for the equal weighted portfolio, which represents the aggregate information of 30 stocks. The estimates of \(m, \Phi\) and \(\Sigma\) are given by

\[
m = \begin{bmatrix} 0.000 \\ 0.021 \\ 0.167 \\ 9.435 \end{bmatrix}, \quad \Phi = \begin{bmatrix} -0.050 & 0.008 & 0.000 & 0.000 \\ -0.385 & 0.555 & 0.002 & -0.001 \\ -2.724 & -0.370 & -0.062 & 0.001 \\ 34.011 & 12.269 & -0.378 & -0.003 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.000 & 0 & 0 & 0 \\ 0 & 0.002 & 0 & 0 \\ 0 & 0 & 3.268 & 0 \\ 0 & 0 & 0 & 88.667 \end{bmatrix}.
\]
(25)
where the maximum eigenvalue of \(\Phi\) is 0.534.

Second, we consider a median case to avoid any outlying impacts which may be caused by the bailout companies such as AIG, Citi and GM. These median values of \(m, \Phi\) and \(\Sigma\) are given by
where the maximum eigenvalue of $\Phi$ is 0.336.

Finally, we consider the most persistent case. With the maximum eigenvalue of $\Phi = 0.6165$, we select the Exxon Mobil Corporation and obtain the following estimates of $m$, $\Phi$ and $\Sigma$:

$$m = \begin{bmatrix} -0.000 \\ 0.058 \\ 0.112 \\ 8.607 \end{bmatrix}, \quad \Phi = \begin{bmatrix} -0.028 & 0.003 & 0.000 & -0.000 \\ -0.280 & 0.343 & 0.003 & -0.002 \\ -2.194 & 0.102 & 0.019 & -0.002 \\ 5.276 & 1.269 & -0.004 & 0.029 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.001 & 0 & 0 & 0 \\ 0 & 0.022 & 0 & 0 \\ 0 & 0 & 2.680 & 0 \\ 0 & 0 & 0 & 75.784 \end{bmatrix} \text{,} \quad (26)$$

Table 3 presents the simulation results in terms of RMSE, MAE, and MAPE of forecasting errors for both short and long positions. Panels A, B, and C report the simulation results for experiments 1, 2, and 3, respectively. The FARVaR models display the smallest forecasting errors for all the experiments, the only exception being MAPE of the third DGP for the long position (see Panel C).\footnote{In this case CAViaR-GARCH achieves the smallest forecasting error (2.41), which is significantly smaller than other models (the average of MAPE is 9.34). But, these findings should be interpreted with more care because it is well-known that MAPE is very sensitive to the small value of $\text{VaR}_t^\alpha$. As gets close to zero, MAPE is likely to be largely inflated in the presence of such outlying observations.} The FARVaR-SIM approach produces the smallest forecasting errors for the long position while the FARVaR-NIG approach displays the smallest ones for the short position. The GARCH models tend to achieve the second best performance. Their forecasting errors are slightly larger than the FARVaR models. Then, overall, RM, HS, FHS, and the CAViaR models follow. The worst forecasting errors are produced by the FEVT models. In sum, the Monte Carlo simulation studies demonstrate that the proposed FARVaR models can produce more accurate forecast of the true $\text{VaR}$ than other existing models.

[TABLE 3 ABOUT HERE]

Though we have established the more satisfactory forecasting performance of the proposed FARVaR models relative to other existing models, we should notice that the DGPs employed in the above simulation experiments are more or less designed to favor the dynamic parametric models over the static nonparametric models by construction. In this regard, it is surprising to find that the FHS models fail to significantly improve their forecasting performance over the HS models. Hence, in next section, we will investigate the
overall performance of all the VaR models through applying the backtesting techniques used to compare the predicted losses from VaR models with the actual losses realised at the end of the period of time.

5 Backtesting the VaR Models

We now turn to examine the performance of the VaR models in terms of the coverage ability, the economic cost and the statistical validity. As strongly recommended by the Basel Committee on Banking Supervision, a backtest is a key part of the internal VaR model development for market risk management. To this end we employ a number of backtesting tools: ECP, BPZ (BCBS, 1996), MRCR (BCBS, 1996), PQL (Koenker and Bassett, 1978), the CC test (Christoffersen, 1998), and the DQ test (Engle and Manganelli, 2004).

The backtesting evaluates coverage ability and economic cost accompanied by the failure of covering the realized extreme event. Hence, all the backtesting tools have been developed based on the failure of the model. The failure is defined by an indication function which takes unity for the case that a realized return is not covered by the VaR forecast:

\[
H_s = 1 \begin{cases} \hat{VaR}_s(\alpha) \end{cases}, \ s = 1, \ldots, N, \tag{28}
\]

where \(\hat{VaR}_s(\alpha)\) is the VaR forecast given the information set available at \(s - 1\) with the nominal coverage probability \(\alpha\).

First, ECP and BPZ evaluate the coverage ability as follows: ECP is calculated by the sample average of \(H_s\), i.e., \(N^{-1} \sum_{s=1}^{N} H_s\). BPZ describes the strength of the internal model through evaluating the failure rate which records the number of daily violations of the 99 percent VaR over the previous 250 business days. One may expect, on average, 2.5 violations under the correct forecasting model. The Basel Committee rules that up to four violations are acceptable, and defines this range as a “Green” zone. If violations are five times or more, the banks fall into “Yellow” (5–9) or “Red” (10+) zone. The penalty is cumulatively imposed by the multiplicative factor \(\kappa\): the multiplicative factor is determined according to the number of violation such as 3 for (0–4 violations), 3.4 (5), 3.5 (6), 3.65 (7), 3.75 (8), 3.85 (9) and 4 (10+), respectively. In the Yellow zone, the supervisor will decide the penalty according on the reason for the violation of the VaR forecast,\(^{14}\) while in the Red zone, the penalty will be automatically generated. Hence, a regulator prefers the VaR model producing ECP close to the nominal probability and BPZ indicating Green zone.

Second, MRCR and PQL evaluate the economic costs associated with the VaR model. Thus the model

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\(^{14}\)See Jorion (2006, p.149) for more detailed descriptions on the categories for the reasons suggested by the Basel Committee: (i) Basic integrity of the model, (ii) model accuracy could be improved, (iii) intraday trading, and (iv) bad luck.
with the minimum MRCR and PQL is preferred by the regulator. Providing that a bank has a sound risk management system and an independent risk-control unit as well as external audits, MRCR is summarized by the following four factors: (i) the quantitative parameters, (ii) the treatment of correlations, (iii) the market risk charge, and (iv) the plus factor (see BCBS, 1996, 2004) for details and Jorion (2006) for a compact summary). Then, MRCR can be formulated by

$$\text{MRCR}_s = \max \left( \kappa \frac{1}{60} \sum_{i=1}^{60} \hat{\text{VaR}}_{s-i} (\alpha), \hat{\text{VaR}}_{s-1} (\alpha) \right) + \text{SRC}_s, \ s = 251, \ldots, N, \quad (29)$$

where $\text{SRC}$ is the additional capital charge for the specific risk (BCBS, 1996, 2004) and $\kappa$ is the Basel penalty factor from BPZ. PQL measures the expected loss of the VaR forecast obtained using the “check” function (Koenker and Bassett, 1978) which can be regarded as a predictive quasi-likelihood (Bertail et al., 2004). It is consistently estimated by

$$\text{PQL} = \frac{1}{N} \sum_{s=1}^{N} (\alpha - H_s) \left( r_s - \hat{\text{VaR}}_s (\alpha) \right). \quad (30)$$

Finally, the CC and DQ tests evaluate the statistical validity of the VaR model. The CC test verifies if the conditional expectation of $H_s$ is equal to the coverage probability. Christoffersen (1998) shows that it is equivalent to testing if the sequence of $H_s$ is identically and independently distributed Bernoulli with the probability $\alpha$. Hence, the suggested LR statistic simultaneously tests if the unconditional coverage probability is $\alpha$ (unconditional coverage test) and if the the binary random variable is independent (independence test). It follows the chi-square distribution with two degrees-of-freedom under the null hypothesis. The DQ test extends the CC test by allowing for more time-dependent information such as lagged realized violations and the VaR forecast. Specifically, we regress the demeaned binary variable on (constant, lagged demeaned binary variable and the VaR forecast), and then test the null hypothesis, $R^2 = 0$ by the Wald test statistic given by

$$\text{DQ} = \left( \underline{\text{H}Z} \right) (Z'Z)^{-1} \left( Z'\underline{\text{H}} \right) = \frac{\hat{\beta}'Z'Z\hat{\beta}}{\alpha (1 - \alpha)} \overset{H_0}{\sim} \chi^2_{p+2}, \quad (31)$$

where $\underline{\text{H}} = (\bar{\text{H}}_{p+1}, \ldots, \bar{\text{H}}_N)'$, $\bar{\text{H}}_s = H_s - \alpha$, $Z = (z_{p+1}, \ldots, z_N)'$ and $z_s = (1, \bar{H}_{s-1}, \ldots, \bar{H}_{s-p}, \hat{\text{VaR}}_s)'$. The regulator prefers the VaR model which is not rejected. We use the first four lags, i.e., $z_s = (1, \bar{H}_{s-1}, \ldots, \bar{H}_{s-4}, \hat{\text{VaR}}_s)'$, where the DQ test statistic follows the chi-squared distribution with 6 degrees-of-freedom under the null.

In next subsections, we repeat the similar rolling estimation procedure as used in Section 4. We estimate
the VaR models using the window size of 250 days over the period, 3 Jan. 2000 - 27 Dec. 2000 and compute one-day-ahead 99% VaR forecasts for 28 Dec. 2000. We repeat this procedure moving forward a day at a time in a rolling manner, and ending with the forecast for 31 Dec. 2008. This generates the 2,013 daily forecasts except for Hewlett-Packard, AT&T, and Verizon Communication which generate 1,428, 2,006 and 1,887 forecasts. We also consider the longer window size (500 days) for investigating any effect of the window size on our evaluations. Furthermore, as the robustness check with respect to the asymmetry of the return distribution, we consider both the long and the short positions.

5.1 Long Position

We apply the VaR models to the left tail behavior of the return distribution with the window size of 250 days. We report ECP, BPZ, MRCR and PQL by averaging the results for the 30 stocks. For the CC and DQ tests, we count the frequency of an individual model being rejected at the 5% significance level out of the 30 stocks.

Panel A of Table 4 presents the backtesting results. In terms of ECP, FARVaR-SIM (FARVaR-NIG) slightly over-forecasts (under-forecasts) VaR, though their outcomes stay in the Green BPZ. Overall, we find that the coverage ability of FARVaR is quite reliable. RM severely under-forecasts VaR, receiving the warning of the Yellow BPZ, a consistent finding with Johansson et al. (1999) and Netftci (2000). This implies that the coverage ability of RM is unreliable so that it cannot be recommended to be used as an internal VaR model. Despite that GARCH employs the (fat-tailed) Student-t distribution, its results are not significantly improved over RM. As expected, the FEVT models substantially over-forecast VaR, though their outcomes stay in the Green zone due to their conservative forecasting (notice under BPZ that over-forecasts of VaR tend to receive better scores given the Basel committee’s prudential principle). Finally, the CAViaR models considerably under-forecast VaR, receiving the warning of Yellow (CAViaR) and Red (CAViaR-GARCH), and they turn out to be the worst models in terms of the coverage ability.

Overall results reveal the number of stylized facts. First, the parametric models tend to under-forecast or over-forecast the tail behavior of the return distribution. Hence, with sufficient sample observations, the nonparametric return distribution could generally improve the coverage ability as clearly demonstrated by the performance of FHS. The coverage ability of the GARCH model can be significantly improved by simulating the tail behavior from the nonparametric empirical distribution. Second, the tail behavior can be more precisely estimated when we model the complex relationships directly among moments or quantiles rather than when we focus on modelling the specific quantile, say at 1 or 5%, like CAViaR. As the FARVaR approach is designed to fully addresses these two important issues, hence it can produce the more reliable
coverage ability than the existing parametric models, RM, GARCH and CAViaR.

Next, we turn to assessing the economic costs accompanied by the VaR models. RM requires the smallest MRCR, followed by GARCH and FARVaR models. On the contrary, the FEVT models demand the most expensive costs as they considerably over-forecast VaR, and thus they have to maintain the large capital, suffering from big opportunity costs. The CAViaR models also demand expensive economic cost to compensate for their poor coverage ability due to their severe under-forecasting. The FARVaR approach is able to reduce the economic costs significantly by producing the reliable coverage ability as well as by explicitly addressing the dynamics of the return distribution.

Finally, we discuss the statistical adequacy of the alternative VaR models by applying the CC test and the DQ test to each of the 30 stocks at the 5% significance level. The columns headed “CC” and “DQ” report a number of the individual model being rejected out of 30 stocks. The null hypothesis of the CC test is rejected for 4 and 6 stocks, respectively, for FARVaR-NIG and FARVaR-SIM models. Notice that these rejection frequencies are well below those of other models such as GARCH (19) and CAViaR (30) and CAViaR-GARCH (30), except for FHS displaying that the null is rejected only once. The rejection frequencies by the DQ test are significantly higher for all the models.

The lowest rejection frequencies are reported at 11 out of 30 for FARVaR-NIG and FGEV, followed by FARVaR-SIM and FHS (12). Again, the null hypothesis of the DQ test is rejected for all stocks when using the CAViaR models. Combining these results, we may conclude that the hybrid semiparametric approach such as FARVaR and FSH is statistically more adequate than either parametric or nonparametric models.

In sum, the performance of the VaR model can be greatly improved by combining relative advantages from parametric and nonparametric models. In this regard, we recommend the proposed FARVaR approach as it turns out to be a promising hybrid approach by substantially improving the coverage ability through avoiding severe misspecification errors and by considerably reducing economic costs thorough explicitly estimating dynamics of the intraday density in a functional space.

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15 A bank manages $1bn and its CFO reports the economic cost using MRCR and PQL on a daily basis. Suppose that the bank’s internal VaR model is FGPD. Then, they should allocate $602.1m against the maximum loss over a 10-day horizon or endure $82.5m loss every day. Suppose that the CFO switch to employ FARVaR-NIG as an internal model. Then, FARVaR-NIG can reduce the capital requirement to $408m and the daily loss to $67.6m. In this regard, FARVaR-NIG can cut down the capital requirement by $194.1m and the daily loss by 14.9m.

16 Berkowitz et al. (2011) conclude that the CC test tends to be less powerful against incorrect VaR models than the the DQ test.
5.2 Short Position

It is not unusual to observe that the return distribution is asymmetric. Hence, it is worth investigating whether the FARVaR approach performs well at the right tail by taking a short position. Panel B of Table 4 presents these backtesting results. Overall results are qualitatively similar to those for the long position. In particular, the coverage ability of FARVaR has somewhat weakened, but its economic cost has slightly reduced. Still, the combined performance of FARVaR is more reliable than those of other models except for FHS.

Next, we find that the individual VaR models are equally or slightly less rejected by both CC and DQ tests, though, for both FGEV and FGPD, the null hypothesis of the CC test is rejected more while the null hypothesis of the DQ test is less rejected. When using CAViaR, the null hypotheses of both tests are rejected for all of 30 stocks. Still, we find that the rejection frequencies of parametric or nonparametric models remain substantially high. This confirms that the hybrid models, FARVaR and FSH, are statistically most robust VaR models, irrespective of the asymmetry of the asset return distribution.

5.3 Longer Window Size

In practice the selection of an optimal window size is a nontrivial issue. As the window size increases, estimation and forecasting precision generally improves. However, it also raises uncertainty about the latent market regimes caused by a sequence of the rare or extreme shocks hitting the market such that it would be more desirable to select the shorter but more homogenous samples rather than longer but heterogeneous ones. In this context we conduct the backtesting exercise using the longer window size of 500 days. The results are qualitatively the same. Results are available on request to authors.

6 Conclusion

The growing importance of intraday activities such as the high-frequency and the algorithm trading in the financial market has motivated us to propose the hybrid FARVaR approach for improving the daily VaR evaluation explicitly using intraday returns. FARVaR combines nonparametric and parametric approaches by applying the dynamic parametric FAR model to forecast nonparametric intraday density. Furthermore, we have provided a more general algorithm of how to construct the daily density directly from the intraday density forecast.

We have conducted comprehensive evaluation exercises using both the simulated and the actual data. First, we find that FAR turns out to be the best functional model predictor of the intraday density function
for all of 30 stocks. Second, via the Monte Carlo simulation studies, we document evidence that \textit{FARVaR} can forecast a true \textit{VaR} more precisely than all other existing models including another hybrid model, FHS. Third, we carry out a number of backtests, finding that \textit{FARVaR} outperform other nonparametric and parametric \textit{VaR} models.

Finally, we highlight a few stylized findings, which may help to enhance our understanding of the contemporary \textit{VaR} analysis. First, as intraday information is helpful in forecasting the daily risk, hence, an accurate modelling of the intraday return density is a key input to improving the daily risk management. In this context we find that a nonparametric density function is constructed to be more robust and suitable for forecasting risks than the parametric counterpart. Second, the \textit{FARVaR} approach is able to accommodate complex dynamics of intraday return density in a flexible manner because \textit{FAR} is a generalization of all classes of autoregressive models. Third, we demonstrate that the hybrid approach can improve the coverage ability and reduce the economic costs, simultaneously. More specifically, the robust nonparametric approach helps to improve the coverage ability while the dynamic parametric modelling in the functional space can reduce the economic costs. Finally, the \textit{FAR} modelling can be easily applied to forecasting other risk measures such as a marginal expected shortfall.

\section*{Acknowledgement}

We are grateful to Heather Anderson, Amir Armanious, Kausik Chaudhuri, Jin-Chuan Duan, Robert Faff, Ana-Maria Fuertes, Matthew Greenwood-Nimmo, Bonsoo Koo, Shuddhasattwa Rafiq, Joon Park, Vasilis Sarafidis, Peter Spencer, Param Sylvapulle, seminar participants at Universities of Brisbane, Bristol, Deakin, Durham, Glasgow, Leeds, Macquarie, Monash, National University of Singapore, York, and conference delegates at 2012 Financial Engineering and Banking Society in London and the ASEM conference in Hobart, July 2014. The second author acknowledges that this paper is an extension of the second chapter of his Ph.D thesis submitted to the University of Leeds in 2011. The usual disclaimer applies.

\section*{References}


Figure 1: Algorithm of FARVaR

**Step 1:** Estimate intraday density functions by a kernel density estimator: \( \hat{f}_t, \hat{f}_{t+1}, \ldots, \hat{f}_T \).

**Step 2:** Model intraday density functions by FAR and forecast it by \( \hat{f}_{t+1} = f + \hat{\lambda}_t (f_t - f) \).

**Step 3:** (1) Daily density function is generated by simulating intraday returns which are randomly drawn from the forecasted density function. (2) Alternatively it is generated by NIG approximation using the first-four moments from the forecasted density function.
Figure 2: Autocorrelation function of moments of intraday returns distribution

(a) volatility

\[ AR(1) = 0.758 \ast \ast \ast \]

(b) skewness

\[ AR(1) = -0.074 \ast \ast \ast \]

Note. 95% CI denotes the 95% confidence interval. *, ** and *** indicate that the null of zero AR(1) coefficient is rejected at 10%, 5% and 1% significance level, respectively.
Figure 3: Contemporaneous correlation patterns among moments of intraday return distribution

(a) mean - volatility

(b) mean - skewness

(c) volatility - skewness

(d) skewness - kurtosis

\[ \text{Corr} (\mu, \sigma | \mu > 0) = 0.672 \]
\[ \text{Corr} (\mu, \sigma | \mu < 0) = -0.633 \]
\[ \text{Corr} (\mu, s) = 0.512 \]
\[ \text{Corr} (\sigma, s | s > 0) = 0.227 \]
\[ \text{Corr} (\sigma, s | s < 0) = -0.366 \]
\[ \text{Corr} (s, k | s > 0) = 0.977 \]
\[ \text{Corr} (s, k | s < 0) = -0.963 \]

Note. \text{Corr} (x, y | z) denotes a correlation coefficient between \( x \) and \( y \) given a condition \( z \). \( \mu, \sigma \) and \( s \) denote mean, volatility and skewness of intraday returns constructed by an equal weighted portfolio of 30 stocks.
Table 1: Descriptive statistics of intraday returns and daily returns

This table presents the descriptive statistics of intraday and daily returns of the 30 stocks listed in DJIA. We calculate mean, standard deviation (Stdev), skewness (SK), and kurtosis (K) of individual each company over the sample period. Panel A report the descriptive statistics of intraday returns. Panel B presents the descriptive statistics of daily returns.

<table>
<thead>
<tr>
<th>Company</th>
<th>Panel A: Intraday Returns</th>
<th>Panel B: Daily Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean (%)</td>
<td>Stdev (%)</td>
</tr>
<tr>
<td>(1) Alcoa Inc.</td>
<td>-0.0007</td>
<td>0.31</td>
</tr>
<tr>
<td>(2) American International Group, Inc</td>
<td>-0.0024</td>
<td>0.54</td>
</tr>
<tr>
<td>(3) American Express Company</td>
<td>-0.0006</td>
<td>0.29</td>
</tr>
<tr>
<td>(4) The Boeing Company</td>
<td>0.0000</td>
<td>0.25</td>
</tr>
<tr>
<td>(5) Citi group Inc.</td>
<td>-0.0010</td>
<td>0.35</td>
</tr>
<tr>
<td>(6) Caterpillar Inc.</td>
<td>0.0003</td>
<td>0.26</td>
</tr>
<tr>
<td>(7) E. L. Du Pont de Nemours &amp; Company</td>
<td>-0.0005</td>
<td>0.23</td>
</tr>
<tr>
<td>(8) Walter Disney Company</td>
<td>-0.0001</td>
<td>0.27</td>
</tr>
<tr>
<td>(9) General Electric Company</td>
<td>-0.0007</td>
<td>0.24</td>
</tr>
<tr>
<td>(10) General Motors</td>
<td>-0.0006</td>
<td>0.28</td>
</tr>
<tr>
<td>(11) Home Depot, Inc.</td>
<td>-0.0003</td>
<td>0.29</td>
</tr>
<tr>
<td>(12) Honeywell International Inc.</td>
<td>0.0006</td>
<td>0.28</td>
</tr>
<tr>
<td>(13) Hewlett-Packard Company</td>
<td>-0.0002</td>
<td>0.23</td>
</tr>
<tr>
<td>(14) International Business Machine Corp.</td>
<td>-0.0010</td>
<td>0.37</td>
</tr>
<tr>
<td>(15) Intel Corporation</td>
<td>0.0001</td>
<td>0.17</td>
</tr>
<tr>
<td>(16) Johnson &amp; Johnson</td>
<td>-0.0008</td>
<td>0.44</td>
</tr>
<tr>
<td>(17) JP Morgan Chase &amp; Co.</td>
<td>-0.0001</td>
<td>0.19</td>
</tr>
<tr>
<td>(18) The Coca-Cola Company</td>
<td>0.0003</td>
<td>0.23</td>
</tr>
<tr>
<td>(19) McDonald’s Corporation</td>
<td>0.0001</td>
<td>0.20</td>
</tr>
<tr>
<td>(20) 3M Company</td>
<td>-0.0003</td>
<td>0.36</td>
</tr>
<tr>
<td>(21) Altria Group Inc.</td>
<td>-0.0005</td>
<td>0.24</td>
</tr>
<tr>
<td>(22) Merck &amp; Co, Inc.</td>
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<td>0.25</td>
</tr>
<tr>
<td>(23) Microsoft Corporation</td>
<td>-0.0003</td>
<td>0.23</td>
</tr>
<tr>
<td>(24) Pfizer, Inc.</td>
<td>0.0001</td>
<td>0.20</td>
</tr>
<tr>
<td>(25) The Procter &amp; Gamble Company</td>
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<td>0.32</td>
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<tr>
<td>(26) AT&amp;T</td>
<td>0.0003</td>
<td>0.23</td>
</tr>
<tr>
<td>(27) United Technology Corporation</td>
<td>-0.0002</td>
<td>0.23</td>
</tr>
<tr>
<td>(28) Version Communication Inc.</td>
<td>-0.0001</td>
<td>0.22</td>
</tr>
<tr>
<td>(29) Wall-Mart Stores, Inc.</td>
<td>0.0004</td>
<td>0.20</td>
</tr>
<tr>
<td>(30) Exxon Mobil Corporation</td>
<td>-0.0018</td>
<td>0.39</td>
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</table>
Table 2: Performance of intraday density forecast

This table presents the performance of intraday density forecast for alternative functional models: FAR, FARFFT, FARWV, AVE, and LAST. For the 30 stocks, one-step ahead rolling forecasting is performed based on the 250 (business days) window size starting at 27/Dec/2000 and ending at 31/Dec/2008. The Hilbert norm ($D_H$), the uniform norm ($D_U$), and the generalized entropy ($D_E$) are employed as divergence criteria (see Eq. (21) for the definition) which are evaluated for the mean value of the 2,013 intraday return density forecasts except for Hewlett-Packard, AT&T and Version Communication Inc. where 1,418, 2,006 and 1,887 forecasts are used for the calculation, respectively. Each figure reports the average of divergence criteria from 30 stocks and (-) indicates the number which a specific model achieves the smallest value for 30 stocks.

<table>
<thead>
<tr>
<th>Models</th>
<th>$D_H$</th>
<th>$D_U$</th>
<th>$D_E$</th>
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<tr>
<td>FAR</td>
<td>0.02852</td>
<td>0.26648</td>
<td>0.04124</td>
</tr>
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<td></td>
<td>(9)</td>
<td>(13)</td>
<td>(4)</td>
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<td>FARFFT</td>
<td>0.02837</td>
<td>0.26625</td>
<td>0.04048</td>
</tr>
<tr>
<td></td>
<td>(14)</td>
<td>(15)</td>
<td>(19)</td>
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<tr>
<td>FARWV</td>
<td>0.02841</td>
<td>0.26781</td>
<td>0.04063</td>
</tr>
<tr>
<td></td>
<td>(7)</td>
<td>(2)</td>
<td>(7)</td>
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<tr>
<td>AVE</td>
<td>0.04568</td>
<td>0.36825</td>
<td>0.05762</td>
</tr>
<tr>
<td></td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
</tr>
<tr>
<td>LAST</td>
<td>0.03415</td>
<td>0.27786</td>
<td>0.05386</td>
</tr>
<tr>
<td></td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
</tr>
</tbody>
</table>
Table 3: Performance of VaR forecast with simulated data

We assume that intraday returns follows the NIG distribution and its first four moments are generated by VAR(1) process (see Assumption 1–4 for details). We generate 78 intraday returns for each day and 251 days for conducting one-step-ahead forecast for both long- and short position based on the past 250 days. We replicate this forecasting exercise 5,000 times with 10 models and evaluate the forecasting precision based on RMSE, MAE, and MAPE (see Eq. (22) for the definition).

<table>
<thead>
<tr>
<th>Criteria</th>
<th>RMSE</th>
<th>MAE</th>
<th>MAPE</th>
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<tr>
<td></td>
<td>Long</td>
<td>Short</td>
<td>Long</td>
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<tr>
<td><strong>Panel A: Simulation 1</strong></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>FARVaR-SIM</td>
<td>1.04</td>
<td>0.98</td>
<td>1.79</td>
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<tr>
<td>FARVaR-NIG</td>
<td>1.25</td>
<td>0.81</td>
<td>1.86</td>
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<td>FHS</td>
<td>1.85</td>
<td>1.41</td>
<td>2.23</td>
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<td>1.83</td>
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<td>2.23</td>
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<tr>
<td>RM</td>
<td>1.26</td>
<td>1.26</td>
<td>1.92</td>
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<td>GARCH</td>
<td>1.15</td>
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<td>1.89</td>
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<tr>
<td>FGEV</td>
<td>3.18</td>
<td>3.25</td>
<td>3.14</td>
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<td>FGPD</td>
<td>3.12</td>
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<td>CAViaR</td>
<td>1.97</td>
<td>2.02</td>
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<td>CAViaR-GARCH</td>
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<td>2.16</td>
<td>2.34</td>
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<td><strong>Panel B: Simulation 2</strong></td>
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<td></td>
<td></td>
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<tr>
<td>FARVaR-SIM</td>
<td>0.90</td>
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<td>2.83</td>
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<td>1.13</td>
<td>0.85</td>
<td>2.88</td>
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<tr>
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<td>3.01</td>
<td>2.26</td>
<td>3.64</td>
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<td>HS</td>
<td>2.98</td>
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<tr>
<td>RM</td>
<td>2.01</td>
<td>2.01</td>
<td>3.17</td>
</tr>
<tr>
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<td>1.78</td>
<td>1.78</td>
<td>3.09</td>
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<td>FGEV</td>
<td>5.27</td>
<td>5.33</td>
<td>5.22</td>
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<td>FGPD</td>
<td>5.15</td>
<td>5.22</td>
<td>5.11</td>
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<tr>
<td>CAViaR</td>
<td>3.29</td>
<td>3.37</td>
<td>3.76</td>
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<tr>
<td>CAViaR-GARCH</td>
<td>3.50</td>
<td>3.56</td>
<td>3.87</td>
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<td><strong>Panel C: Simulation 3</strong></td>
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<td></td>
<td></td>
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<tr>
<td>FARVaR-SIM</td>
<td>1.10</td>
<td>1.01</td>
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<td>FARVaR-NIG</td>
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<td>0.94</td>
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<td>2.79</td>
</tr>
<tr>
<td>HS</td>
<td>2.32</td>
<td>1.78</td>
<td>2.79</td>
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<td>RM</td>
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<td>1.62</td>
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<tr>
<td>FGEV</td>
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<td>4.07</td>
<td>3.95</td>
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<tr>
<td>FGPD</td>
<td>3.93</td>
<td>4.01</td>
<td>3.87</td>
</tr>
<tr>
<td>CAViaR</td>
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<td>2.68</td>
<td>2.83</td>
</tr>
<tr>
<td>CAViaR-GARCH</td>
<td>2.65</td>
<td>2.78</td>
<td>2.90</td>
</tr>
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</table>
Table 4: Backtesting VaR models with 250 window size

This table presents the backtesting results by calculating the average values of the 30 stocks for ECP, BPZ, MRCR, and PQL, and counting the number which a specific model is rejected at the 5% significance for the 30 stocks (the CC and DQ tests). Backtestings for each stock are evaluated using 2,013 daily VaR forecasts while Hewlett-Packard Company, AT&T and Version Communication Inc. are calculated using 1,428, 2,006 and 1,887 daily VaR forecasts, respectively.

<table>
<thead>
<tr>
<th>Models</th>
<th>ECP</th>
<th>BPZ</th>
<th>MRCR</th>
<th>PQL</th>
<th>CC</th>
<th>DQ</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Long Position</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>FARVaR-SIM</td>
<td>0.87%</td>
<td>Green</td>
<td>42.65%</td>
<td>6.79%</td>
<td>6</td>
<td>12</td>
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<tr>
<td>FARVaR-NIG</td>
<td>1.07%</td>
<td>Green</td>
<td>40.80%</td>
<td>6.76%</td>
<td>4</td>
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</tr>
<tr>
<td>HS</td>
<td>1.19%</td>
<td>Green</td>
<td>45.28%</td>
<td>7.44%</td>
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<tr>
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<td>1.10%</td>
<td>Green</td>
<td>45.29%</td>
<td>6.87%</td>
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<tr>
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<td>1.82%</td>
<td>Yellow</td>
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<tr>
<td>GARCH</td>
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<td>FGEV</td>
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<td>Green</td>
<td>58.22%</td>
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<td>FGPD</td>
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<td><strong>Panel B: Short Position</strong></td>
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<td>39.74%</td>
<td>5.98%</td>
<td>5</td>
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<td>FARVaR-NIG</td>
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<td>20</td>
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<td>11</td>
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<td>52.53%</td>
<td>6.60%</td>
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<td>6</td>
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<tr>
<td>FGPD</td>
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<td>Green</td>
<td>53.77%</td>
<td>6.73%</td>
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<td>Yellow</td>
<td>42.70%</td>
<td>7.30%</td>
<td>30</td>
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