An Estimated DSGE Model: Explaining Variation in Nominal Term Premia, Real Term Premia, and Inflation Risk Premia*

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Abstract

This paper develops a DSGE model which is shown to explain variation in the nominal and real term structure as well as inflation surveys and four macro variables for the UK economy. The model is estimated based on a third-order approximation to allow for time-varying term premia. We find a fall in nominal term premia during the 1990's which mainly is caused by lower inflation risk premia. A structural decomposition further shows that this fall is driven by negative preference shocks, lower fixed production costs, positive investment shocks, and a more aggressive response to inflation by the Bank of England.

Keywords: Market price of risk, Non-linear filtering, Quantity of risk, Epstein-Zin-Weil preferences, Third-order perturbation.

JEL: C51, E10, E32, E43, E44.

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1 Introduction

The nominal term structure reveals expectations about future one-period nominal interest rates and investors' compensation for uncertainty related to interest rates with longer maturities. This compensation is typically referred to as the nominal term premium. The expected nominal one-period interest rate may further be decomposed into a real interest rate and expected inflation. Similarly, the nominal term premium can be split into the real term premium and the inflation risk premium. Decomposing term structure data in this way is often very useful for monetary policy. For instance, a central bank pursuing inflation targeting may only want to raise the policy rate in response a steepening of the nominal yield structure if this increase is caused by higher inflation expectations and not if it is due to higher nominal term premia.

A large number of papers have used reduced-form term structure models to study nominal and real interest rates and their term premia.[1] However, little is currently known about the structural determinants behind the dynamics of these term premia. The contribution of the present paper is to close this gap in the literature by estimating a Dynamic Stochastic General Equilibrium (DSGE) model to carry out a structural decomposition and interpretation of nominal term premia, real term premia, and inflation risk premia. We use a New Keynesian DSGE model with Epstein-Zin-Weil preferences, capital accumulation, stochastic and deterministic trends, sticky prices, and a central bank controlling monetary policy based on a Taylor-rule. To allow for time-variation in term premia, the model is solved to third order by the perturbation method. Using UK data, the model is estimated by non-linear filtering methods to match the nominal and real term structure, inflation surveys, and four macro variables.

To provide the structural interpretation of term premia, we need to overcome a number of challenges. Firstly, it is in general difficult for DSGE models to reproduce the dynamics of the nominal term structure (see for instance Haan (1995) and Rudebusch & Swanson (2008)). Matching this aspect of the data is clearly a necessary first step for a reliable decomposition of the information content in term structure data.

Secondly, the mechanisms driving the nominal and real term structure impose substantial requirements on the stochastic discount factor and the DSGE model in general. Broadly speaking, in order to match the real term structure the model should generate levels of future consumption that correspond to the average expectations of investors. The model is also required to generate inflation expectations in line with the expectations held by the average investor in order to fit the nominal term structure. Furthermore, we also require that the model reproduces observed time series for consumption and inflation along with inflation expectations from surveys. Indeed, specifying a structural model with these properties is a major challenge.

[1] A non-exhaustive list includes the work by Barr & Campbell (1997), Evans (1998), Evans (2003), Ang, Bekaert & Wei (2008), D'Amico, Kim & Wei (2009), Christensen, Lopez & Rudebusch (2010), Hordahl & Tristani (2010), and Joyce, Lildholdt & Sorensen (2010).

Thirdly, the solution to DSGE models must be approximated with non-linear terms in order to generate time-varying term premia, but such approximations are quite time consuming to compute. For instance, Rudebusch & Swanson (2008) report that it takes about 10 minutes to solve a third-order approximation to their benchmark model. If the model is estimated, hundreds of thousands of function evaluations are necessary and a new model solution must be computed for every evaluation.

Finally, the existing literature uses a normality assumption or a second-order approximation of the stochastic discount factor to decompose the information content in the nominal and real term structure. We cannot apply this decomposition as our model is approximated to third order. Hence, an extension of the current method to decompose information embodied in term structure data is required.

In an empirical application, the suggested model is estimated on UK data after 1992 when the current inflation targeting regime was initiated. Our focus on the UK economy is motivated by the presence of a large and liquid market for real bonds. We highlight the following results. Firstly, the model delivers in general a satisfying fit to the two term structures while simultaneously matching inflation expectations and four macro variables. The only exception is the 1-quarter nominal interest rate where larger model errors are encountered. In total, we match 17 time series using just 7 structural shocks. Secondly, and as in much of the finance literature, we find a reduction in nominal term premia immediately after inflation targeting was adopted in 1992 and again after the Bank of England became operational independent in 1997. Nominal term premia then remain low and stable until 2006 where these premia start to increase. In general, most of the variation in nominal term premia relates to inflation risk premia within our model. Thirdly, a decomposition of the 10-year inflation risk premium shows that favorable preference and investment shocks contribute to a permanently lower level of inflation risk from 1992 to 2008. Upward pressure on the 10-year inflation risk premium during this period is mainly related to a rise in fixed production costs, possible due to higher oil and gas prices, and this explains the increase in inflation risk after 2006. We also find that a more aggressive monetary policy to inflation lowers inflation risk premia after inflation targeting in 1992 and operational independence to the Bank of England in 1997. Fourthly, adopting the typical terminology from the finance literature, our model implies a gradual reduction in the market price of inflation risk during the 1990's. We also find that the quantity of inflation uncertainty falls after inflation targeting is adopted in 1992 and operational independence to the Bank of England in 1997. Finally, our estimated model implies a fairly volatile 10-year nominal term premium with a standard deviation of 83 basis points. This model property is obtained by relying on a high risk-aversion for the household through Epstein-Zin-Weil preferences.

The rest of this paper is organized as follows. Section 2 presents our New Keynesian DSGE model which is extended with the nominal and real term structure in Section 3. It is also shown in Section 3 how the information content in these term structures can be used to extract nominal term premia, real term premia, and inflation risk premia. Section 4 discusses how the solution to our model is approximated by a third-order perturbation approach. We estimate the model on UK data and conduct the structural decomposition of term premia in Section 5. Concluding comments are provided in Section 6.

2 The DSGE model

This section presents a DSGE model with the same basic structure as in Smets & Wouters (2007) and Altig, Christiano, Eichenbaum & Linde (2011). We next describe the behavior of the three types of agents in this economy: i) households, ii) firms, and iii) a central bank.[2]

2.1 The households

The behavior of the households is described by a representative agent with Epstein-Zin-Weil preferences following the work of Epstein & Zin (1989) and Weil (1990). These preferences have recently been introduced into DSGE models by Rudebusch & Swanson (2012) and Binsbergen, Fernandez-Villaverde, Koijen & Rubio-Ramirez (2010). Applying the same specification as in Rudebusch & Swanson (2012), the value function V_t for the household is given by

$$V_t \equiv \begin{cases} u_t + \beta \left(E_t \left[V_{t+1}^{1-\xi_3} \right] \right)^{\frac{1}{1-\xi_3}} & \text{for } u_t \geq 0 \\ u_t - \beta \left(E_t \left[(-V_{t+1})^{1-\xi_3} \right] \right)^{\frac{1}{1-\xi_3}} & \text{for } u_t < 0 \end{cases} \tag{1}$$

Here, E_t denotes the conditional expectation given information available at time t and $\beta \in]0, 1[$ is the household's subjective discount factor. The parameter $\xi_3 \in \mathbb{R} \backslash \{1\}$ controls the degree of relative risk-aversion, where higher values of ξ_3 imply higher levels of risk-aversion for $u_t \geq 0$, and vice versa for $u_t < 0$. The benefit of Epstein-Zin-Weil preferences is that they allow us to disentangle the household's risk-aversion and intertemporal elasticity of substitution which are closely linked when using standard power preferences. By allowing for a high level of risk-aversion, Rudebusch & Swanson (2012) show that these preferences help an otherwise standard DSGE model match the dynamics of the 10-year nominal term premium.

We assume that the value of the periodic utility index u_t is determined by consumption c_t and labor supply h_t in the following way

$$u_t \equiv \frac{d_t}{1-\xi_2} \left[\left(\frac{c_t - bc_{t-1}}{z_t^*} \right)^{1-\xi_1} (1 - h_t)^{\xi_1} \right]^{1-\xi_2}, \tag{2}$$

[2] As in Ravn (1997), Nelson & Nikolov (2004), DiCecio & Nelson (2007), among others, the UK economy is here modelled as a closed economy.

where $\xi_1 \in]0,1[$ and $\xi_2 \in]0,1[\cup]1,\infty[$. Non-seperability between consumption and labor is introduced for two reasons. Firstly, this assumption makes the utility index non-negative for $\xi_2 \in]0,1[$ and non-positive for $\xi_2 \in]1,\infty[$, and the specification therefore fits nicely with (1). Secondly, non-seperability is consistent with the presence of a balanced growth-path in the economy as shown by King, Plosser & Rebelo (1988).

The parameter $b \in [0,1]$ controls the degree of external habit formation in consumption. We introduce this feature because various habit specifications in general improve the ability of DSGE models to reproduce certain macroeconomic and financial moments (see for instance Campbell & Cochrane (1999), Fuhrer (2000), Christiano, Eichenbaum & Evans (2005), and Hordahl, Tristani & Vestin (2008)). It is further assumed that the household's utility from consumption is measured in deviation from the consumption trend z_t^* which we specify in the following section.[3]

Following Smets & Wouters (2007), Justiniano & Primiceri (2008), and others, we include preference shocks d_t by letting

$$\ln(d_{t+1}) = \rho_d \ln(d_t) + \epsilon_{d,t+1}, \tag{3}$$

where $\rho_d \in]-1,1[$. The error terms $\{\epsilon_{d,t}\}_{t=1}^{\infty}$ are assumed to be independent and normally distributed, i.e. $\epsilon_{d,t} \sim \mathcal{NID}\left(0, \sigma_d^2\right)$.

The consumption good is constructed from a continuum of differentiated goods ($c_{i,t}$, $i \in [0,1]$) and the aggregation function

$$c_t \equiv \left[\int_0^1 c_{i,t}^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}}. \tag{4}$$

Hence, the demand for good i is

$$c_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} c_t, \tag{5}$$

where $P_t \equiv \left[\int_0^1 P_{i,t}^{1-\eta} di\right]^{1/(1-\eta)}$ is the nominal price index. The continuously compounded inflation rate π_t is then $e^{\pi_t} = P_t/P_{t-1}$.

The household's real period-by-period budget constraint is given by

$$E_t D_{t,t+1} x_{t+1}^h + c_t + i_t/\left(e_t \Upsilon_t\right) = x_t^h/e^{\pi_t} + w_t h_t + r_t^k k_t + \phi_t. \tag{6}$$

Expenditures are allocated to i) state-contingent claims $\left(E_t D_{t,t+1} x_{t+1}^h\right)$, ii) consumption ($c_t$), and iii) investment $(i_t/\left(e_t \Upsilon_t\right))$. The variable $D_{t,t+1}$ denotes the nominal stochastic discount factor. We follow Greenwood, Hercowitz & Krusell (1997) in specifying the investment expenditures and allow for a time-varying real price of investment in terms of the consumption good $1/\left(e_t \Upsilon_t\right)$. Change in this relative

[3] This feature resembles the "ratio" habit specification considered by Abel (1990) which is introduced in addition to the habit level determined by b. Scaling consumption by z_t^* leaves the utility index and the value function untrended, implying that $\beta < 1$ ensures a finite value function if u_t is bounded (see Epstein & Zin (1989)).

price are modelled exogenously, and $1/\left(e_t \Upsilon_t\right)$ can therefore be interpreted as investment shocks. These shocks are assumed to evolve according to a stationary AR(1) process around a deterministic trend. We include the deterministic trend to allow for the common empirical property that the mean investment growth is higher than the average growth rate in consumption and output. More formally, we let

$$\ln e_{t+1} = \rho_e \ln e_t + \epsilon_{e,t+1}, \tag{7}$$

where $\rho_e \in \left]-1,1\right[$, $\epsilon_{e,t} \sim \mathcal{NID}\left(0, \sigma_e^2\right)$, and $\ln \Upsilon_t = \ln \Upsilon_{t-1} + \ln \mu_{\Upsilon,ss}$.

The right hand side of (6) describes the household's total wealth in period t. It consists of i) pay-off from state-contingent claims purchased in period $t-1$ $\left(x_t^h / e^{\pi_t}\right)$, ii) real labor income $(w_t h_t)$, iii) return from selling capital services to firms $\left(r_t^k k_t\right)$, and iv) dividend payments received from firms (ϕ_t). The latter are restricted to zero in steady state.

The household owns the capital stock k_t in the economy and therefore also makes the investment decision. When doing so, it is constrained by the law of motion for capital

$$k_{t+1} = (1-\delta) k_t + i_t \left(1 - \frac{\kappa}{2}\left(\frac{i_t}{\Upsilon_t z_t^* i_{SS}} - 1\right)^2\right), \tag{8}$$

where $\delta \in [0,1]$ is depreciation and $\kappa \geq 0$. Following Christiano et al. (2005), we allow for investment adjustment costs but in this paper relate these costs to the balanced growth-path of investment, i.e. $\Upsilon_t z_t^* i_{SS}$, instead of i_{t-1}.

2.2 The firms

There is a continuum of firms, each supplying a differentiable good $y_{i,t}$ using

$$y_{i,t} \equiv \begin{cases} k_{i,t}^{\theta} \left(a_t z_t h_{i,t}\right)^{1-\theta} - \psi_t z_t^* & \text{if } k_{i,t}^{\theta} \left(a_t z_t h_{i,t}\right)^{1-\theta} - \psi_t z_t^* > 0 \\ 0 & \text{else} \end{cases} \tag{9}$$

with $\theta \in [0,1]$ and $\psi_t \geq 0$. The variables $k_{i,t}$ and $h_{i,t}$ denote the amount of capital and labor used by firm i, respectively. Technology shocks are allowed to have a stationary component a_t and a non-stationary component z_t. We include the traditional stationary technology shocks because Hordahl et al. (2008) and Rudebusch & Swanson (2012) find that they are important in order to generate sizable nominal term premia in the US. The non-stationary technology shocks are primarily introduced to explain the mean growth rate of consumption and output, and a large fraction of the cyclical variation in these time series. Formally, we let

$$\ln a_{t+1} = \rho_a \ln a_t + \epsilon_{a,t+1}, \tag{10}$$

where $\rho_a \in]-1,1[$ and $\epsilon_{a,t} \sim \mathcal{NID}\left(0,\sigma_a^2\right)$. The non-stationary component is given by

$$\ln\left(\frac{\mu_{z,t+1}}{\mu_{z,ss}}\right) = \rho_z \ln\left(\frac{\mu_{z,t}}{\mu_{z,ss}}\right) + \epsilon_{z,t+1} \tag{11}$$

where $\mu_{z,t} \equiv z_t/z_{t-1}$, $\rho_z \in]-1,1[$, and $\epsilon_{z,t} \sim \mathcal{NID}\left(0,\sigma_z^2\right)$. The innovations $\epsilon_{a,t}$ and $\epsilon_{z,t}$ are assumed to be mutually independent and so are all other innovations in the model.

Following Altig et al. (2011), the value of z_t^* is defined to be equal to $\Upsilon_t^{\theta/(1-\theta)} z_t$ and z_t^* can therefore be interpreted as an overall measure of technological progress in the economy. As in Altig et al. (2011), we scale ψ_t in (9) by z_t^* to ensure the presence of a balanced growth-path in the model.

Smets & Wouters (2007) document the importance of real supply shocks specified as shocks to firms' markup rates. However, with Calvo price contracts, these markup shocks prevent an exact recursive representation of the equilibrium conditions which is needed for a non-linear approximation to our model. Instead, we introduce real supply shocks by letting firms' fixed production costs be time-varying beyond the variation in z_t^*. The inclusion of these real supply shocks can be motivated by variation in firms' fixed production costs due to changes in oil prices, maintenance costs, firms' subsidies, etc. We therefore let

$$\ln\left(\frac{\psi_{t+1}}{\psi_{ss}}\right) = \rho_\psi \ln\left(\frac{\psi_t}{\psi_{ss}}\right) + \epsilon_{\psi,t+1}, \tag{12}$$

where $\rho_\psi \in]-1,1[$ and $\epsilon_{\psi,t+1} \sim \mathcal{NID}\left(0,\sigma_\psi^2\right)$.

Firms are assumed to maximize the present value of their nominal dividend payments given by

$$div_{i,t} \equiv E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \phi_{i,t+l}, \tag{13}$$

where $\phi_{i,t}$ is real dividend payments from firm i in period t. The firm faces a number of constraints when maximizing $div_{i,t}$. The first is related to the good produced by firm i, as firms must satisfy demand in all markets. In aggregate terms this implies

$$y_t = c_t + i_t/\left(e_t \Upsilon_t\right). \tag{14}$$

The second constraint is a real budget restriction which give rise to the expression for real dividend payments from firm i in period t

$$\phi_{i,t} = \left(P_{i,t}/P_t\right) y_{i,t} - r_t^k k_{i,t} - w_t h_{i,t}. \tag{15}$$

The first term in (15) denotes real revenue from sales of the i'th good. The firm's expenditures are allocated to purchases of capital services $\left(r_t^k k_{i,t}\right)$ and payments to workers $\left(w_t h_{i,t}\right)$.

The third constraint introduces sticky prices following Calvo (1983). Here, a fraction $\alpha \in [0,1[$ of randomly chosen firms cannot set the optimal nominal price of the good they produce in each period. Instead, these firms set current prices equal to prices in the previous period, i.e. $P_{i,t} = P_{i,t-1}$. Using the law of large numbers, we therefore have

$$1 = (1-\alpha)\,\tilde{p}_t^{1-\eta} + \alpha \exp(\pi_t)^{\eta-1} \tag{16}$$

where $\tilde{p}_t$ denotes the real optimal price for all optimizing firms in period t.

Finally, the aggregate resource constraint is derived by summing output for all firms. Using the property that $k_{i,t}/h_{i,t}$ is constant for each firm within our model, it follows that

$$k_t^{\theta}\,(a_t z_t h_t)^{1-\theta} - z_t^* \psi_t = y_t s_t, \tag{17}$$

where $h_t \equiv \int_0^1 h_{i,t} di$, $k_t \equiv \int_0^1 k_{i,t} di$, and $s_t \geq 1$ captures the resource inefficiencies from price stickiness (see Schmitt-Grohe & Uribe (2007)). The law of motion for s_t is given by

$$s_t = (1-\alpha)\,\tilde{p}_t^{-\eta} + \alpha \exp\{\pi_t\}^{\eta}\, s_{t-1}. \tag{18}$$

Hence, effects of steady state inflation for macro variables and interest rates are captured via the state variable s_t. We also note that the inclusion of s_t is in general needed to obtain a higher-order accurate approximation to our model, even in the case of no steady state inflation, i.e. when $\pi_{ss} = 0$.

2.3 The central bank

We approximate the behavior of the central bank by a rule for the one-period continuously compounded interest rate $r_{t,1}$

$$r_{t,1} = \rho_r r_{t-1,1} + (1-\rho_r)\left(r_{ss,1} + \beta_{\pi,t}\,(\pi_t - \pi_{ss}) + \beta_y \ln\left(\frac{y_t}{y_{ss} z_t^*}\right)\right) + \epsilon_{r,t}, \tag{19}$$

where $\epsilon_{r,t} \sim \mathcal{NID}\left(0, \sigma_{\epsilon_r}^2\right), \rho_r \geq 0$, $\beta_{\pi,t} \geq 0$, and $\beta_y \geq 0$. That is, the central bank aims to close the inflation gap $(\pi_t - \pi_{ss})$ and the output gap $\ln(y_t/(y_{ss} z_t^*))$, while at the same time smoothing changes in the policy rate. Note that we follow Justiniano & Primiceri (2008), Rudebusch & Swanson (2012), among others, and define the output gap by output in deviation from its balanced growth-path $y_{ss} z_t^*$. An interesting feature in (19) is that we allow the coefficient for the inflation gap $\beta_{\pi,t}$ to be time-varying, implying that the policy response to a given inflation gap may change over time. Such changes should affect all nominal interest rates and may be important in order to understand their dynamics. As in Fernandez-Villaverde, Guerron-Quintana & Rubio-Ramirez (2010), $\beta_{\pi,t}$ is modelled exogenously according to

$$\ln\left(\frac{\beta_{\pi,t+1}}{\beta_\pi}\right) = \rho_{\beta_\pi} \ln\left(\frac{\beta_{\pi,t+1}}{\beta_\pi}\right) + \epsilon_{\beta_\pi,t+1} \tag{20}$$

where $\rho_{\beta_\pi} \in \left]-1,1\right[$ and $\epsilon_{\beta_\pi,t+1} \sim \mathcal{NID}\left(0, \sigma^2_{\beta_\pi}\right)$.[4] Due to rational expectations in our model, households and firms are aware of this potential change in monetary policy and take it into account when making their decisions.[5]

3 The nominal and real term structure

This section derives the nominal and real term structure from the micro-founded stochastic discount factor and no-arbitrage arguments. Using a third-order approximation to allow for time-varying term premia, we then show how the difference between the nominal and real term structure can be decomposed into i) expected future inflation, ii) a convexity term, and iii) inflation risk premia.

3.1 Deriving the nominal and real term structure

The presence of state contingent claims implies that we can price all financial assets in the economy based on standard no-arbitrage arguments. Hence, the price of a zero-coupon bond maturing n periods into the future and paying one unit of cash at maturity is

$$P_{t,n} = E_t\left[D^{real}_{t,t+n} \prod_{j=1}^{n} \frac{1}{e^{\pi_{t+j}}}\right], \tag{21}$$

where $D^{real}_{t,t+n} \equiv \beta\lambda_{t+n}/\lambda_t$ is the real stochastic discount factor and λ_t denotes the household's marginal value of income. In our case, it holds that

$$D^{real}_{t,t+1} = \beta \frac{d_{t+1}}{d_t} \frac{u_c\left(\frac{c_{t+1}-bc_t}{z^*_{t+1}}, 1-h_{t+1}\right)}{u_c\left(\frac{c_t-bc_{t-1}}{z^*_t}, 1-h_t\right)} \left[\frac{\left(E_t\left[(V_{t+1})^{1-\xi_3}\right]\right)^{\frac{1}{1-\xi_3}}}{V_{t+1}}\right]^{\xi_3} \frac{1}{\mu_{z^*,t+1}} \tag{22}$$

when $u_t \geq 0$.[6] The first term $\beta\frac{d_{t+1}}{d_t}$ is the household's subjective discount factor adjusted for preference shocks. The second term is the familiar ratio of future marginal utility of consumption to the present value of marginal utility, where the utility might depend on consumption habits and leisure. The third term (in the squared brackets) is present because of the Epstein-Zin-Weil preferences in our model,

[4] It is easy to see that shocks to $\beta_{\pi,t}$ do not have first-order effects. Accordingly, the standard requirements for a stable unique equilibrium also hold with (19).

[5] Another possibility would be to let $\beta_{\pi,t}$ be regime depended as in Davig & Leeper (2007), Farmer, Waggoner & Zha (2010), and Ferman (2010). However, obtaining the non-linear model solution in this case is very challenging and we therefore prefer the specification in (20).

[6] for $u_t < 0$, then V_{t+1} is replaced by $-V_{t+1}$ in (22).

i.e. $\xi_3 \neq 0$, and amplifies the effect of unexpected changes in the household's wealth as measured by the value function. This function summarizes current and future utility, and we therefore have that expectations to all future levels of habit adjusted consumption and leisure affect bond prices when $\xi_3 \neq 0$. The last term $1/\mu_{z^*,t+1}$ is due to the presence of trends in the economy where $\mu_{z^*,t} \equiv z_t^*/z_{t-1}^*$.

With continuous compounding, it holds that $e^{-nr_{t,n}} = P_{t,n}$ where $r_{t,n}$ is the nominal interest rate in period t for a bond maturing in n periods. The nominal term structure is then derived by calculating these interest rates for different values of n.

Similarly, the price of a real zero-coupon bond maturing n periods into the future and paying one unit of consumption at maturity is

$$P_{t,n}^{real} = E_t\left[D_{t,t+n}^{real}\right]. \tag{23}$$

All real interest rates at different maturities $r_{t,n}^{real}$ are then derived from $e^{-nr_{t,n}^{real}} = P_{t,n}^{real}$.

3.2 Decomposing the difference between nominal and real interest rates

We begin by the standard decomposition of the nominal term structure

$$r_{t,n} = \frac{1}{n}E_t\left[\textstyle\sum_{j=1}^{n} r_{t+j-1,1}\right] + C_{t,n} + TP_{t,n}, \tag{24}$$

where $C_{t,n}$ is the convexity term and $TP_{t,n}$ refers to term premium. A similar decomposition holds for the real term premium $TP_{t,n}^{real}$ and its related convexity term $C_{t,n}^{real}$. For the subsequent decomposition, let $\Delta^n\hat{\lambda}_{t+n} \equiv \hat{\lambda}_{t+n} - \hat{\lambda}_t$ denote the n'th difference for the household's marginal value of income where $\hat{\lambda}_t \equiv \ln(\lambda_t/\lambda_{ss})$. We also introduce $\hat{\Pi}_{t+n} \equiv \sum_{i=1}^{n}(\pi_{t+i} - \pi_{ss})$ as accumulated inflation from period t to period $t+n$ when expressed in deviation from π_{ss}. Based on this parsimonious notation, Appendix A derives expressions for nominal and real interest rates accurate up to third order and thus extend the results in Hordahl et al. (2008) from a second-order to a third-order approximation. The difference between nominal and real interest rates define break-even inflation rates which are given by

$$r_{t,n} - r_{t,n}^{real} = \frac{1}{n}\left(E_t\left[\hat{\Pi}_{t+n}\right] + C_{t,n}^{infl} + TP_{t,n}^{infl}\right). \tag{25}$$

The first component $E_t\left[\hat{\Pi}_{t+n}\right]$ is expected inflation until expiry of the zero-coupon bond. The second component $C_{t,n}^{infl}$ is an inflation convexity term which is given by

$$C_{t,n}^{infl} \equiv -\frac{1}{2}\textstyle\sum_{i=1}^{n} Var_t(\pi_{t+i}) - \frac{1}{6}\left(\left(E_t\left[\hat{\Pi}_{t+n}\right]\right)^3 - E_t\left[\left(\hat{\Pi}_{t+n}\right)^3\right] + 3E_t\left[\hat{\Pi}_{t+n}\right]E_t\left[\left(\hat{\Pi}_{t+n}\right)^2\right]\right). \tag{26}$$

Note that $C_{t,n}^{infl}$ in a second-order approximation reduces to the familiar expression $-\frac{1}{2}\sum_{i=1}^{n} Var_t(\pi_{t+i})$. The third component $TP_{t,n}^{infl}$ in (25) is inflation risk premia

$$\begin{aligned} TP_{t,n}^{infl} \equiv\; & -\sum_{i=0}^{n-1}\sum_{k=1+i}^{n} Cov_t(\pi_{t+1+i}, \pi_{t+k}) + Cov_t\left(\Delta^n \hat{\lambda}_{t+n}, \hat{\Pi}_{t+n}\right) \\ & +\frac{1}{2}Cov_t\left(\left(\Delta^n \hat{\lambda}_{t+n}\right)^2, \hat{\Pi}_{t+n}\right) - \frac{1}{2}Cov_t\left(\left(\hat{\Pi}_{t+n}\right)^2, \Delta^n \hat{\lambda}_{t+n}\right) \\ & +\frac{1}{2}\left(Cov_t\left(\Delta^n \hat{\lambda}_{t+n}, \hat{\Pi}_{t+n}\right) - 3E_t\left[\Delta^n \hat{\lambda}_{t+n}\hat{\Pi}_{t+n}\right]\right)\left(E_t\left[\Delta^n \hat{\lambda}_{t+n}\right] - E_t\left[\hat{\Pi}_{t+n}\right]\right) \end{aligned} \tag{27}$$

which simplifies to the first two terms in (27) at second order. An important implication of (25) is that the difference between nominal and real interest rates also in the case of a third-order approximation can be decomposed into three components which conceptually resemble the content of any term structure. As a result, the difference between the nominal and real term structure defines an inflation term structure.

A general feature of all Gaussian affine term structure models and DSGE models approximated up to second order is that inflation risk premia equal the difference between nominal and real term premia. A proof is provided in Appendix B. This also holds in our model if

$$C_{t,n}^{real} + C_{t,n}^{infl} = C_{t,n}, \tag{28}$$

because the expectation of one-period nominal interest rates, real interest rates, and inflation always adds up with continuous compounding. For comparability with the existing literature, we impose (28) such that our model also implies

$$TP_{t,n}^{infl} = TP_{t,n} - TP_{t,n}^{real}. \tag{29}$$

In practice, nominal term premia are computed using the approach in Rudebusch & Swanson (2012), i.e. $TP_{t,n}$ is the difference between $r_{t,n}$ and the yield-to-maturity on the corresponding risk-neutral bond where payments are discounted by $r_{t,1}$ instead of the stochastic discount factor. A similar procedure is used to compute $TP_{t,n}^{real}$, and equation (29) then gives $TP_{t,n}^{infl}$.

4 A non-linear approximated model solution

It is well-established that a third-order approximation around the deterministic steady state allows for variation in risk premia as implied by the exact, but infeasible, solution to our model. However, the presence of non-stationary variables and the size of the model complicate computing a third-order approximation by the perturbation method. We deal with the issues of stationarity and the size of the model in turn.

4.1 Inducing stationarity

The presence of trends in investment shocks Υ_t and technology shocks z_t implies that some variables are non-stationary - for instance consumption c_t, output y_t, and investment i_t. This fact must be taken into account when using the perturbation method because it only gives reliable results when the economy is close to the approximation point, and the existence of non-stationary variables clearly violates this requirement. We deal with this complication by adopting the standard procedure where all non-stationary variables are scaled by their cointegrating factor to make them stationary. For instance, c_t and y_t are scaled by $1/z_t^*$ and i_t is scaled by $1/(\Upsilon_t z_t^*)$, implying that $C_t \equiv c_t/z_t^*$, $Y_t \equiv y_t/z_t^*$, and $I_t \equiv i_t/(\Upsilon_t z_t^*)$ are stationary. Using this equivalent representation of the model, the standard perturbation method can be applied.

4.2 A third-order perturbation approximation

In the considered model, it is fairly standard to derive i) market clearings conditions, ii) first-order conditions for the representative household, iii) first-order conditions for firms, and iv) recursive equations for the nominal and real term structure. The exact solution to the system is given by

$$\mathbf{y}_t = \mathbf{g}\left(\mathbf{x}_t, \sigma\right) \tag{30}$$

$$\mathbf{x}_{t+1} = \mathbf{h}\left(\mathbf{x}_t, \sigma\right) + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \tag{31}$$

where σ is the perturbation parameter and $\boldsymbol{\epsilon}_{t+1}$ contains the structural innovations. The vector $\mathbf{y}_t$ with dimension $n_y \times 1$ includes all control variables while the state vector $\mathbf{x}_t$ with dimension $n_x \times 1$ denotes pre-determined variables. Both $\mathbf{y}_t$ and $\mathbf{x}_t$ are expressed in deviation from the steady state.[7] The functions $\mathbf{g}\left(\mathbf{x}_t, \sigma\right)$ and $\mathbf{h}\left(\mathbf{x}_t, \sigma\right)$ are unknown but can be approximated by polynomials in $(\mathbf{x}_t, \sigma)$ around the deterministic steady state as shown by Judd & Guu (1997). We use the codes by Schmitt-Grohé & Uribe (2004) to compute the first-order and second-order derivatives of $\mathbf{g}\left(\mathbf{x}_t, \sigma\right)$ and $\mathbf{h}\left(\mathbf{x}_t, \sigma\right)$. The third-order derivatives are computed using the codes accompanying Andreasen (forthcoming). Based on the work by Kim, Kim, Schaumburg & Sims (2008) and Andreasen, Fernandez-Villaverde & Rubio-Ramirez (2012), we apply the pruning method when setting up the state space system for the approximated model solution.

Although the perturbation method is computationally fast compared to many other approximation methods, the size of the model makes it numerically challenging to find non-linear terms for $\mathbf{g}\left(\mathbf{x}_t, \sigma\right)$

[7] Focusing on linear innovations in the state equation is without loss of generality as the state vector can be extended to account for any non-linearities between $\mathbf{x}_t$ and $\boldsymbol{\epsilon}_{t+1}$ (see Andreasen (forthcoming)). Our model only contains linear innovations, and it is therefore unnecessary to introduce the extended state vector which would only make the approximation process more time consuming.

and $\mathbf{h}(\mathbf{x}_t, \sigma)$.[8] However, this numerical problem has recently been made easier to solve by Andreasen & Zabczyk (2010) who develop an efficient method for computing bond prices in DSGE models. They propose a two-step perturbation method where the output from the first perturbation step is used as input in a second perturbation step. In the first step, the DSGE model without bond prices beyond one period is solved up to any desired order by the standard perturbation method. The second step then perturbates the fundamental pricing equation for bond prices up to the same order. Andreasen & Zabczyk (2010) then show that derivatives of bond prices can be solved for in a recursive manner, given the output from the first perturbation step. Only simple summations are needed to compute these bond prices which therefore can be computed almost instantaneously. As emphasized by Andreasen & Zabczyk (2010), this "perturbation on perturbation" (POP) method gives *exactly* the same expression for bond prices as standard perturbation, which solves simultaneously for all bond prices and all other variables in the model.

When taking the model to the data, we also include long-term inflation expectations. However, including the 10-year inflation expectations in a quarterly model adds 40 additional control variables to the system. We want to avoid such a large extension of the system for numerical reasons, and we therefore show in Appendix C that expected values of any control variable in DSGE models solved up to third order can also be computed by the POP method in a fast and straightforward manner.[9]

As a result, the POP method enables us to solve the model to third order in just 5 seconds on a standard desktop computer and therefore makes estimation feasible.

5 An application to the UK economy

This section estimates our DSGE model on UK data. We begin by describing the data and our estimation methodology in Section 5.1 and 5.2, respectively. The estimation results are reported in Section 5.3, and the model's ability to fit the data is examined in Section 5.4. Impulse response functions and a variance decomposition are presented in Section 5.5, before we conduct a structural decomposition of term premia in Section 5.6.

[8] Our model has 11 state variables and 10 control variables without the two term structures. In a quarterly model with a maximum maturity of 10-years, the two term structures add $2 \times 40 = 80$ control variables to the system. Hence, we have $n_y = 90$. To compute the second-order terms $\mathbf{g_{xx}}$ and $\mathbf{h_{xx}}$, we need to solve a linear system with a dimension of $(n_y + n_x)\, n_x(n_x + 1)/2 = 6,666$. Solving this system typically requires a lot of computer memory and is very time consuming even though this system is sparse. The numerical problem to find the third-order terms $\mathbf{g_{xxx}}$ and $\mathbf{h_{xxx}}$ is even more challenging because it requires solving a linear system with a dimension of $(n_y + n_x)\, n_x(n_x + (n_x - 1)(n_x - 2)/6) = 28,888$.

[9] These formulas may also be of useful when computing impulse response functions.

5.1 UK data

The model is estimated using nominal interest rates, real interest rates, inflation surveys, and four macro series. We next describe each of these data sources in turn. The nominal term structure is represented by the 1-quarter, 1-year, 3-year, 5-year, 7-year, and 10-year constant maturity interest rates on government bonds. All these interest rates are measured at the end of the quarter and expressed in annual terms. The data is available from the Bank of England's homepage, except the 1-quarter rate where we use the implied rate from a 3-month Treasury bill.[10]

The real term structure is represented by the 3-year, 5-year, 7-year, and 10-year constant maturity rates on index-linked government bonds.[11] The interest rates are from end of quarter and provided by the Bank of England. We also note that the series for the 3-year real interest rate is incomplete with missing values from 1995 Q4 to 1996 Q4 and from 2005 Q1 to 2005 Q2. The next section discusses how to account for these missing values in our estimation procedure.

Data on inflation surveys is available from Consensus Forecasts which provides inflation expectations for the retail price index (RPI) at different horizons.[12] We focus on expected inflation 1 and 3 years ahead as well as long term expected inflation given by the average expectation 5 to 10 years ahead (i.e. the 5-year 5-year forward inflation expectation).[13]

The last group of variables consists of four macro variables. The first series is the inflation rate in RPI which is used instead of the more familiar consumer price index (CPI) because our real interest rates are derived from bonds index-linked to the RPI. The remaining macro variables are the real growth rates in consumption, investment, and GDP.[14]

5.2 Our estimation methodology

Let the vector $\mathbf{y}_t^{obs}$ contain the 17 data series used for the estimation. We allow for measurement errors in $\mathbf{y}_t^{obs}$ and assume that these errors $\mathbf{w}_t$ are of the form $\mathbf{w}_t \sim \mathcal{NID}(\mathbf{0}, \mathbf{R_w})$ where $\mathbf{R_w}$ is a diagonal matrix. To economize on the number of free parameters in $\mathbf{R_w}$, it is further assumed that 20% of the variation in the series for inflation and the three real growth rates are due to measurement errors.[15] Expressed in annual terms, this implies measurement errors with standard deviations of: i) 17 basis points for inflation, ii) 24 basis points for consumption growth, iii) 76 basis points for investment

[10] The study by Lildholdt, Panigirtzoglou & Peacock (2007) adopt a similar approach and verifies that this 3-month interest rate is in line with the data from the Bank of England.

[11] The 1-quarter and 1-year real interest rates are not available in this data set.

[12] Only inflation expectations on RPIX (that is RPI excluding mortgage interest payments) are available from 1997 and onwards. However, the difference between RPI and RPIX is in general small.

[13] The setup of the Consensus survey implies that inflation expectations with a horizon of 11 and 13 quarters are used to approximate the inflation expectations at a horizon of 3 years.

[14] The growth rate in consumption is calculated from real final consumption expenditures. We use the series for real gross fixed capital formation to calculate the growth rate in investment. The growth rate in GDP is calculated from real GDP. These variables are seasonal adjusted and downloaded from Datastream. All series are computed as annual growth rates and expressed in per capita based on the total population in the UK.

[15] An & Schorfheide (2007) use a similar assumption.

growth, and iv) 24 basis points for output growth. This assumption can be considered as restricting the model, in a probabilistic sense, to match the four macro series.

The size of the measurement errors in the two term structures and the inflation surveys are left as free parameters in order to assess the model's ability to fit these variables. We adopt the following parsimonious specification for the variance of the measurement errors along the nominal term structure (Var_n)

$$\ln \sqrt{Var_n} = \gamma_1 + \gamma_2 n + \gamma_3 n^2, \tag{32}$$

where n denotes the maturity of the interest rates. For the variance of the measurement errors in the real term structure (Var_n^{real}), we let

$$\ln \sqrt{Var_n^{real}} = \ln \gamma_4 + \ln \sqrt{Var_n}. \tag{33}$$

Finally, the standard deviations of measurement errors for inflation expectations at 1 year $\left(\sigma_{\pi_4^e}\right)$, 3 years $\left(\sigma_{\pi_{12}^e}\right)$, and 5 to 10 years $\left(\sigma_{\pi_{long}^e}\right)$ are estimated as free parameters.

The set of structural parameters in our model is partitioned into two groups. The first group contains coefficients which are hard to identify in aggregated macro time series and are therefore determined based on standard calibration arguments. The second group consists of all the remaining parameters which are estimated. We emphasize that this partitioning of the structural parameters is standard practice when taking large DSGE models to the data (see Christiano et al. (2005), Smets & Wouters (2007), Justiniano & Primiceri (2008), among others).

We now describe how parameters in the first group are determined. For firms' production function, we set θ to a standard value of 0.36 (see for instance Ravn (1997)). For a given value of θ, the average real growth rate in consumption (0.0219) and average investment growth (0.0278) can be used to find the deterministic trends in technology shocks $\mu_{z,ss}$ and investment shocks $\mu_{\Upsilon,ss}$. This follows from the fact that the mean of consumption and investment growth are given by

$$E\left[4 \ln \mu_{c,t}\right] = 4 \ln \left(\mu_{\Upsilon,ss}^{\frac{\theta}{1-\theta}} \mu_{z,ss}\right) \tag{34}$$

$$E\left[4 \ln \mu_{i,ss}\right] = 4 \ln \left(\mu_{\Upsilon,ss}^{\frac{1}{1-\theta}} \mu_{z,ss}\right). \tag{35}$$

This calibration implies $\mu_{z,ss} = 1.0046$ and $\mu_{\Upsilon,ss} = 1.0015$. The depreciation rate is set to $\delta = 0.025$, and the parameter η controlling firms' steady state markup is calibrated to 4.33 based on an assumption of a 30% markup. Finally, the steady state inflation rate π_{ss} is calibrated to match the average inflation rate in the sample using the non-linear calibration technique outlined in Andreasen (2011).

All the remaining parameters are estimated by Quasi Maximum Likelihood (QML) based on the Central Difference Kalman Filter (CDKF) developed by Norgaard, Poulsen & Ravn (2000). This filter extends the standard Kalman filter to non-linear and non-normal state space systems where the non-linear moments in the filtering equations are approximated at least up to second-order accuracy. Andreasen (2010*b*) shows that this QML estimator can be expected to be consistent and asymptotically normal for DSGE models solved up to third-order. The main advantage of this QML estimator is that it is very fast to compute even for models approximated to third order. This is convenient in our case with 17 observables and a fairly large state vector.[16]

The presence of missing values for some inflation surveys and the 3-year real interest rate complicate the evaluation of the quasi log-likelihood function, and the existing algorithm for the CDKF no longer applies. It is, however, straightforward to show that the standard method to deal with missing observations in the Kalman filter (see for instance Durbin & Koopman (2001)) also applies to the CDKF. That is, we only need to adjust the dimension of the Kalman gain in the CDKF and the one-step ahead prediction density for the observables to match the available data points in each period. All other steps in the CDKF are unaffected by the presence of missing values and are as given in Norgaard et al. (2000).

5.3 The estimated structural parameters

The model is estimated on data from 1992 Q3 to 2008 Q2. The starting date of 1992 Q3 is chosen for two reasons. Firstly, the UK introduced monetary policy with inflation targeting in this quarter, and this is the key assumption underlying our interest rate rule in (19). Secondly, Bianchi, Mumtaz & Surico (2009) find evidence of a regime change for the interaction between the macroeconomy and the nominal term structure in 1992 Q3.

The estimated structural parameters and their standard errors are reported in Table 1.[17] The household is seen to place a reasonable weight on leisure in the utility index ($\xi_1 = 0.63$) and displays a moderate degree of habit formation ($b = 0.29$). With $\xi_2 = 2.39$, these estimates imply an intertemporal elasticity of substitution of 0.47 in steady state. The parameter related to the Epstein-Zin-Weil preferences ξ_3 is estimated to be -183, and this gives strong preferences for early resolution of uncertainty. A measure for the level of relative risk aversion is $(\xi_2+\xi_3(1-\xi_2))/(1-b)$ according to Swanson (forthcoming). Using this measure we find a very high relative risk-aversion of 336, and our results are in this sense similar to those of Rudebusch & Swanson (2012). As they point out, there are two likely explanations for this finding. Firstly, high risk-aversion in the case of Epstein-Zin-Weil preferences is similar to a setup where households have low risk-aversion but account for model uncertainty as illustrated by Barillas, Hansen & Sargent (2009). Secondly, Malloy, Moskowitz & Vissing-Jørgensen (2009) show that variability in consumption for stockholders in the US is higher than the variation in

[16] An alternative to the CDKF is to use particle filtering and approximate the likelihood function as in Fernández-Villaverde & Rubio-Ramírez (2007). Unfortunately, particle filtering is computationally infeasible for our model, given its dimension and relative high approximation order.

[17] The optimization of the quasi log-likelihood function is done with a modified version of the CMA-ES routine which Andreasen (2010*a*) shows can optimize likelihood functions for DSGE models.

aggregate consumption, and risk-aversion is therefore estimated to be substantial lower for stockholders when compared to a representative agent using aggregate consumption. In other words, the high risk-aversion we find is likely to compensate for insufficient consumption variation within our model, partly because we do not account for model uncertainty and partly because consumption variability for bondholders is likely to be higher than implied by aggregate consumption. We also note that the large Epstein-Zin-Weil parameter only affects the model's second and third-order terms. This implies that it has a large impact on various risk premia but not on the dynamics of key macrovariables which mainly are determined by the model's first-order terms.

Table 1 also shows that we find sizeable adjustment costs in investment ($\kappa = 5.40$) and fairly sticky prices with $\alpha = 0.79$. The latter implies that the average firm approximately change its prices once every year. The central bank focuses mostly on stabilizing inflation ($\beta_\pi = 1.49$) compared to output $\left(\beta_y = 0.05\right)$. In doing so it assigns a large weight to smoothing changes in the policy rate as $\rho_r = 0.88$. Both findings are standard for the UK economy (DiCecio & Nelson (2007), Harrison & Oomen (2010)).

Given the estimated values, our non-linear calibration of the steady state inflation implies $\pi_{ss} = 1.0157$. This is the value of π_{ss} which ensures that the model reproduces the mean level of RPI inflation (1.0069 expressed in quarterly terms) when accounting for the household's precautionary saving motive.

< Table 1 about here >

5.4 Model fit

Figure 1 shows historical time series (the black lines) and model-implied time series (the red lines) for all 17 variables. Starting with the nominal term structure, we observe some differences between data and model-implied series for the 1-quarter and 1-year rates. A better fit is obtained for all other nominal interest rates where the model closely matches the historical series. Inflation expectations are shown in the third row of Figure 1, and we see that the model successfully matches the gradual fall in inflation expectations during the 1990's. This is particularly clear for long-term inflation expectations.

The model is also largely able to fit the real term structure as shown by the second part of Figure 1. Notable errors are only visible at the 3-year maturity around 2000 where the real rate is predicted to be slightly lower than what is observed in the data. The reduced-form affine model by Joyce et al. (2010) experiences similar problems, and the authors attribute it to i) the opening of a real bond market in the US and ii) the introduction of the Minimum Funding Requirements in the UK which increased the demand for real bonds among UK pension funds. We also note that the two periods of missing observations for the 3-year real interest rate are well accounted for by our model as the reappearance of this rate does not induce abnormal model errors.

The remaining charts show that the model at the same time is able to reproduce the dynamics of RPI inflation and the three real growth rates.

< Figure 1 about here >

In summary, our model delivers a generally satisfying fit to the data and should therefore serve as a useful framework for a structural decomposition and interpretation of term premia. In this context it should also be noted that we fit 17 time series with just 7 structural shocks, whereas the norm in much empirical macro is to use the same number of shocks as the number of observables (see for instance Smets & Wouters (2007)).

5.5 Analyzing the model

Before we turn to the decomposition and interpretation of the various term premia, it is instructive to analyze how shocks affect the model and which shocks are most important for matching the data. We deal with each of these issues in turn.

5.5.1 Impulse response functions

The impulse response functions for the 7 shocks are shown in Figure 2 where each row shows responses to the same shock for a selected number of variables.[18] Given the topic of the paper, we focus on explaining the economics behind impulse response functions for term premia.

We start by considering the effects of a positive shock to firms' fixed costs (ψ_t) in the first row of Figure 2. Inflation is here seen to increase which is due to the following effects. The increased fixed costs make it harder for firms to meet demand, and they therefore require more labor and capital. Getting access to more labor coincides with households desire because they receive lower dividends from firms and therefore see consumption fall due to a negative wealth effect. The response of households is therefore to off-set some of the reduction in consumption by increasing their labor supply, even at a lower wage (not shown). The second and dominating effect for inflation relates to firms' demand for additional capital, where investment adjustment costs make it relatively expensive for households to increase investment and hence capital. To ensure that a higher investment level is profitable for households, we therefore see a higher price of capital r_t^k (not shown) which increases marginal costs mc_t via firms' optimality condition for capital, $mc_t\theta\left(a_t z_t h_t/k_t\right)^{1-\theta} = r_t^k$. Through the sticky prices, the higher marginal costs then drive up prices and inflation.

The response of the central bank is to increase the short nominal interest rate, and this leads to a higher 10-year nominal interest rate as shown in top left chart of Figure 2. The rise in inflation is higher than the rise in nominal interest rates and this generates lower real interest rates as shown in

[18]To account for non-linearities in the model solution, impulse responses for $\mathbf{y}_t$ are defined as $IR\left(y_{t+l}\right) \equiv E_t\left(\mathbf{y}_{t+l}; \mathbf{x}_t, \{\tilde{\boldsymbol{\epsilon}}_{t+i}\}_{i=1}^{l}\right) - E_t\left(\mathbf{y}_{t+l}; \mathbf{x}_t, \{\boldsymbol{\epsilon}_{t+i}\}_{i=1}^{l}\right)$ where $\tilde{\boldsymbol{\epsilon}}_{t+1} = \boldsymbol{\epsilon}_{t+1} + 1$ and $\tilde{\boldsymbol{\epsilon}}_{t+i} = \boldsymbol{\epsilon}_{t+i}$ for $i = 0, 1, 2, 3, ..., l$. The expectations are evaluated by simulation. Here, $\boldsymbol{\epsilon}_{t+i}$ are drawn from its specified distribution while each value of $\mathbf{x}_t$ is taken from a sufficient long simulated sample for the state variables.

the top chart, second to the left. Hence, the value of the 10-year real bond increases as consumption falls, and this bond can therefore be used as a hedge by households to generate a more smooth consumption profile. It is thus desirable for households to buy the 10-year real bond, and the 10-year real term premium is therefore positively correlated with consumption. In contrast, the value of the 10-year nominal bond falls as consumption decreases, and this asset therefore makes it more difficult for households to generate a smooth consumption profile. Accordingly, households require a premium for holding the 10-year nominal bond, and we therefore see an increase in the 10-year nominal term premium as $TP_{t,40} = TP^{infl}_{t,40} + TP^{real}_{t,40}$.

The same logic can be used to explain the variation in term premia following other shocks. In the interest of space, we simply emphasize two features in relation to the remaining impulse response functions. Firstly, the different responses to a non-stationary technology shock (z_t) compared to a stationary technology shock (a_t) relate mainly to the positive wealth effects following an increase in z_t as noted by Rudebusch & Swanson (2012). That is, a rise in z_t makes it desirable for households to work less and enjoy more leisure. To maintain the required production level, firms therefore have to increase the wage level to partly off-set this effect, and this raises production costs which in return leads to higher inflation and higher nominal interest rates. This wealth effect is less pronounced after a stationary technology shock which therefore lowers inflation and all nominal interest rates.

The second set of impulse responses which we highlight relate to a temporary change in monetary policy to a more aggressive inflation reaction, i.e. a rise in $\beta_{\pi,t}$. This change in policy lowers the variation in inflation and other macro variables which in turn reduces the risk faced by households. The response of the risk-averse households is therefore to lower their amount of precautionary saving which causes a small boom and higher inflation in the economy.[19] The central bank reacts by increasing the short interest rate, while we see a small fall in the 10-year nominal interest rate. Hence, households experience an increase in consumption and a rise in the price of the 10-year nominal bond, and this asset therefore makes it more difficult for households to maintain a smooth consumption profile. As a result, households require compensation for holding this bond, and the 10-year nominal term premium is therefore negatively correlated with consumption. This explains why our model generates a reduction in nominal term premia following a more aggressive reaction to inflation by the central bank.

$<$ Figure 2 about here $>$

5.5.2 Variance decomposition

In linearized DSGE models, the importance of various shocks for different dimensions of the data are usually addressed by a shock or variance decomposition. However, the non-linear terms in our model

[19] This effect is not present in a second-order approximation to the model where the degree of precausionary saving is constant. In a second-order approximation, we find that the model implies a very small contraction in consumption, investment, and output following this shock.

solution complicates such a decomposition because the structural shocks enter in a non-linear fashion. We overcome this problem by linearizing the non-linear model solution around the estimated states to get a locally linear state space system. Based on this approximation, the standard method for a variance decomposition can then be applied.[20] Results from the variance decomposition are provided in Table 2.

Starting with a decomposition of the short-term dynamics (1-step ahead), we see that the nominal term structure is mainly explained by preference shocks, the two technology shocks, and monetary policy shocks. It is interesting to note that monetary policy is the key driver of variation in the real term structure which is possible due to the presence of sticky prices in the model. Given the large impact of monetary policy on real interest rates, we also see that monetary policy explains about 30% of the short-term variation in consumption and output growth. As for inflation, its dynamics is equally determined by preference shocks, stationary technology shocks, and monetary policy.

The long-term dynamics of the data are examined by increasing the forecast horizon for the decomposition to 8 periods. We see that preference shocks are the key driver of the nominal term structure, and the two types of technology shocks now explain a smaller fraction of the variation in the data. The real term structure is explained by several disturbances; only non-stationary technology and investment shocks appear unimportant. Investment shocks are not surprisingly a key determinant of investment growth, but also of long-term inflation expectations which almost exclusively are explained by this shock.

< Table 2 about here >

5.6 A structural decomposition and interpretation of term premia

The aim of this section is to use the estimated model for a structural decomposition and interpretation of term premia. As a starting point for this analysis, Section 5.6.1 displays model-implied estimates of nominal and real term premia along with inflation risk premia. The following two subsections contain two decompositions of term premia.

5.6.1 Historical estimates of term premia

The estimated time series for nominal term premia at different maturities are shown in the top chart of Figure 3.[21] We first note that nominal term premia fall immediately after the introduction of inflation targeting in 1992 Q3. At the 10-year maturity, the premium drops from 110 to 60 basis points between 1992 Q3 and 1993 Q4. During the next three years, nominal term premia are seen to gradually increase and are in 1997 Q1 close to the level at the start of our sample. After operational independence to

[20] We refer to the paper's technical appendix for additional details.

[21] All risk premia in this paper are for the corresponding spot interest rates.

the Bank of England in 1997 Q2, nominal term premia fall yet again and reach an all-time low of 40 basis points at the 10-year maturity in 1998 Q3. This corresponds to a total fall of 60 basis points after the Bank of England became operational independent. With the exception of the period from 1999-2000, nominal term premia then remain at the new low level until 2006. It is interesting to note that these findings are broadly similar to those in Joyce et al. (2010) who use an reduced-form affine term structure model. After 2006, we once again see an increase in nominal term premia, and at the end of our sample they have again returned to the level of 1992 Q3.

The following two charts in Figure 3 show estimated time series for real term premia and inflation risk premia to explore which of the two components drive nominal term premia. We first note that real premia have a fairly low and stable level throughout the estimation period and therefore do not contribute much to the variation in nominal premia. Ang et al. (2008) reach the same conclusion for the US, whereas Joyce et al. (2010) find more variation in real term premia in the UK. Inflation risk premia, on the other hand, are very volatile and display broadly the same pattern as nominal term premia. Note in particular how inflation risk premia fall sharply after the adoption of inflation targeting in 1992 Q3 and again after operational independence to the Bank of England in 1997 Q2. Similarly, the increase in nominal term premia after 2006 is generated by a fairly sharp increase in inflation risk premia.

< Figure 3 about here >

Another observation from Figure 3 is that nominal term premia are quite volatile. Indeed, the standard deviation for the 10-year nominal term premium is 83 basis points. We emphasize that it is usually difficult to obtain such a volatile nominal term premium in DSGE models without compromising the model's ability to fit the macroeconomy as argued by Rudebusch & Swanson (2008). We match the macro economy as shown by Figure 1, and the flexible nominal term premia is obtained by relying on a very high level of risk-aversion through the Epstein-Zin-Weil preferences.

Given that most of the variation in nominal term premia relates to inflation risk, we focus on inflation risk premia in the remaining part of this section. In the interest of space, we consider the 10-year maturity but the subsequent decompositions could easily be done for other risk premia and at different maturities.

5.6.2 Structural decomposition of the 10-year inflation risk premium

The structural foundation of our model allows us to decompose risk premia further because we can assess how much of the variation is generated by each of the structural shocks. For instance, did monetary policy reduce inflation risk premia after 1992 and 1997 or was it due to a sequence of favorable shocks to the UK economy? Similarly, what explains the recent rise in inflation risk after 2006?

Effects of the individual structural shocks on the 10-year inflation risk premium are examined by considering the evolution of this premium when only one shock is present, while all other shocks are restricted to their values at the start of the sample. Such counter-factuals allow us to explore what the inflation risk premium would have been if the UK economy only had experienced that particular shock. Results from these counter-factuals are shown in Figure 4. Here we note that four shocks account for most of the dynamics in the 10-year inflation risk premium, namely i) preference shocks, ii) shocks to firms' fixed production costs, iii) investment shocks, and iv) shocks to the central bank's inflation reaction. The time series for the structural shocks are plotted in Figure 5.

Combining Figure 4 and 5, we make the following observations. Firstly, a sequence of negative preference shocks from 1997 to 1999 reduces the inflation risk premium by nearly 80 basis points. Some of this reduction is off-set by positive preference shocks from 1999-2004, implying that the overall effect from these shocks is a fall of 50 basis points in the inflation risk premium during the considered period. A widely used structural interpretation of these preference shocks is that they correspond to demand shocks. The impulse response functions in Figure 2 are consistent with this interpretation as a positive preference shock gives higher inflation and a temporary consumption boom, leaving less resources for household saving via lower investment. Accordingly, negative demand shocks originating from within the household account for a sizable decrease in the inflation risk premium during the considered time period.

Secondly, a reduction in firms' fixed production costs in the beginning of the 1990's accounts for a fall of 40 basis points in the inflation risk premium. The picture is reversed after 1998 where higher production costs generate an increase in the inflation risk premium of about 60 basis points from 1998-2000. Even more striking is the effect from these shocks after 2006 where they account for a further increase in the inflation risk premium of 60 basis points. Accordingly, shocks to firms' fixed production costs explain a rise in the inflation risk premium of nearly 80 basis points from 1992 Q3 to the end of our sample. The overall pattern in these costs broadly resembles the evolution in UK energy prices during the period as ψ_t is positively correlated with oil prices (0.63) and gas prices (0.50). Thus, our model suggests that these price increases account for a large fraction of the variation in the inflation risk premium.

Thirdly, a sequence of positive investment shocks from 1995 to 2005 lowers inflation risk gradually by about 40 basis points. Recall that these shocks enter in the household's budget constraint as $i_t / (e_t \Upsilon_t)$ and thereby lower the real cost of investing.

Fourthly, changes in the central bank's reaction to inflation also account for a large fraction of the variation in inflation risk. In particular, the Bank of England is found to be more aggressive in stabilizing inflation after inflation targeting in 1992 Q3 and its operational independence in 1997 Q2, with both events leading to a reduction in the inflation risk premium of about 30 and 50 basis points, respectively. Hence, to the extent that changes in monetary policy are well-approximated by $\beta_{\pi,t}$, our model suggests that good policy explains some of the reduction in the inflation risk premium during the 1990's. Having established its credibility in the 1990's, the Bank of England is found to adopt a less aggressive attitude to inflation after 2000 which raises the inflation risk premium. Overall, the impact on inflation risk from monetary policy is therefore seen to balance out at the end of our sample, and changes to monetary policy does not contribute to a permanent fall in the inflation risk premium from 1992 Q3 to 2008 Q2.

< Figure 4 about here >

< Figure 5 about here >

To summarize, we find that favorable preference and investment shocks contribute to a permanently lower level of the inflation risk premium. Upward pressure on inflation risk during our sample period is mainly related to a rise in fixed production costs, possible due to higher oil and gas prices, and this explains the increase in the inflation risk premium after 2006. Monetary policy is found to temporarily lower the inflation risk premium during the 1990's, in particular following regime changes in 1992 and 1997. However, our model suggests that monetary policy is somewhat looser after 2000 and therefore does not contribute to a permanent fall in the inflation risk premium from 1992 to 2008.

5.6.3 Market price of nominal risk versus quantity of inflation risk

The 10-year inflation risk premium can also be decomposed based on the standard finance terminology where this premium is expressed in terms of "market price of nominal risk" and "quantity of inflation risk". The first term reveals the required compensation for carrying additional inflation risk when buying nominal bonds instead of real bonds, whereas the second term illustrates the uncertainty linked to future inflation. We use the standard measure for the market price of nominal risk $\sqrt{Var_t\left[D_{t,t+1}\right]}/E_t\left[D_{t,t+1}\right]$ where $D_{t,t+1}$ is the nominal stochastic discount factor (see Hansen & Jagannathan (1991), Cogley & Sargent (2008), among others). This implies that the inflation risk premium can be decomposed as

$$TP_{t,n}^{infl} \equiv \underbrace{\frac{\sqrt{Var_t\left[D_{t,t+1}\right]}}{E_t\left[D_{t,t+1}\right]}}_{\text{Market price of nom. risk}} \times \underbrace{\frac{E_t\left[D_{t,t+1}\right]TP_{t,n}^{infl}}{\sqrt{Var_t\left(D_{t,t+1}\right)}}}_{\text{quantity of inflation risk}},$$

where the second term defines the quantity of inflation risk at maturity n. Note here that the time-variation in the quantity of inflation risk is endogenously generated by the model and does not stem from our shock specifications which all have constant conditional second moments. In other words, our model does not rely on shocks with time-dependent second moments (i.e. stochastic volatility or GARCH effects) to generate time-variation in the quantity of inflation risk.

The top chart in Figure 6 shows a more or less gradual fall in the market price of nominal risk from 1992 to 2005. This finding is similar to the results in the affine and homoscedastic reduced-form model by Joyce et al. (2010) where all variation in risk premia is due to changes in the market price of risk. A repricing of risk occurs after 2005 where the market price of risk increases steadily, and it is by the end of our sample close to the level of the mid 1990's.

The time series for the quantity of inflation risk is provided in the bottom chart of Figure 6. As expected, we see a fall in inflation uncertainty surrounding the adoption of inflation targeting in 1992 Q3 and operational independence to the Bank of England in 1997Q2. These events are followed by a steady rise in inflation uncertainty from 1999-2002, after which the uncertainty level stabilized somewhat. Overall, the evolution in the quantity of inflation risk suggests that the fall in inflation risk from 1992 to 2005 mainly stems from a lower market price of nominal risk in our model.

< Figure 6 about here >

6 Conclusion

This paper develops a DSGE model which explains variation in the nominal and real term structure along with key macro variables for the UK economy. The proposed model belongs to the New Keynesian tradition and incorporates Epstein-Zin-Weil preferences to generate sizable and time-varying term premia when solved up to third order. With the exception of the 1-quarter nominal interest rate, our model delivers a satisfying fit to the two term structures while simultaneously matching inflation expectations and four macro variables. We find a fall in nominal term premia after the introduction of inflation targeting in 1992 and operational independence to the Bank of England in 1997. Nominal term premia then remain low and stable until 2006 when these premia start to increase. In general, most of the variation in nominal term premia relates to changes in inflation risk in our model. A structural decomposition shows that favorable preference and investment shocks contribute to a permanently lower level of the inflation risk premium from 1992 to 2008. Upward pressure on inflation risk during this period is mainly related to a rise in fixed production costs, possible due to higher oil and gas prices, and this explains the increase in inflation risk after 2006. Monetary policy is found to lower the inflation risk premium after the adoption of inflation targeting in 1992 and operational independence to the Bank of England in 1997. However, our model suggests that monetary policy is somewhat looser after 2000 and therefore does not contribute to a permanent fall in inflation risk premia from 1992 to 2008. We also find a gradual reduction in the market price of inflation risk during the 1990's, while the quantity of inflation uncertainty falls after inflation targeting in 1992 and operational independence to the Bank of England in 1997.

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A The difference between nominal and real interest rates

This section derives third-order approximated expressions for real and nominal interest rates using the same method as in Hordahl et al. (2008). A third-order approximation to the expression for a real bond price is given by

$$P^{real}_{ss,n}\left(1+\hat{p}^{real}_{t,n}+\tfrac{1}{2}\left(\hat{p}^{real}_{t,n}\right)^2+\tfrac{1}{6}\left(\hat{p}^{real}_{t,n}\right)^3\right)=E_t\left[D^{real}_{ss,n}\left(1+\hat{d}^{real}_{t,t+n}+\tfrac{1}{2}\left(\hat{d}^{real}_{t,t+n}\right)^2+\tfrac{1}{6}\left(\hat{d}^{real}_{t,t+n}\right)^3\right)\right]$$

$$\Updownarrow$$

$$\hat{p}^{real}_{t,n}+\tfrac{1}{2}\left(\hat{p}^{real}_{t,n}\right)^2+\tfrac{1}{6}\left(\hat{p}^{real}_{t,n}\right)^3=E_t\left[\hat{d}^{real}_{t,t+n}+\tfrac{1}{2}\left(\hat{d}^{real}_{t,t+n}\right)^2+\tfrac{1}{6}\left(\hat{d}^{real}_{t,t+n}\right)^3\right]$$

where $\hat{p}^{real}_{t,n}=\ln\left(P^{real}_{t,n}/P^{real}_{ss,n}\right)$ and $\hat{d}^{real}_{t,t+n}=\ln\left(D^{real}_{t,t+n}/D^{real}_{ss,ss+n}\right)$. We need third-order accurate expressions for $\left(\hat{p}^{real}_{t,n}\right)^2$ and $\left(\hat{p}^{real}_{t,n}\right)^3$ based on $P^{real}_{t,n}=E_t\left[D^{real}_{t,t+n}\right]$. These expressions are given by

$$\left(\hat{p}^{real}_{t,n}\right)^2=\left(E_t\left[\hat{d}^{real}_{t,n}\right]\right)^2+E_t\left[\hat{d}^{real}_{t,n}\right]E_t\left[\left(\hat{d}^{real}_{t,n}\right)^2\right]$$

and

$$\left(\hat{p}^{real}_{t,n}\right)^3=\left(E_t\left[\hat{d}^{real}_{t,n}\right]\right)^3$$

Hence,

$$\begin{aligned}
\hat{p}^{real}_{t,n}&=E_t\left[\hat{d}^{real}_{t,t+n}+\tfrac{1}{2}\left(\hat{d}^{real}_{t,t+n}\right)^2+\tfrac{1}{6}\left(\hat{d}^{real}_{t,t+n}\right)^3\right]-\tfrac{1}{2}\left(\hat{p}^{real}_{t,n}\right)^2-\tfrac{1}{6}\left(\hat{p}^{real}_{t,n}\right)^3\\
&=E_t\left[\hat{d}^{real}_{t,t+n}+\tfrac{1}{2}\left(\hat{d}^{real}_{t,t+n}\right)^2+\tfrac{1}{6}\left(\hat{d}^{real}_{t,t+n}\right)^3\right]-\tfrac{1}{2}\left(\left(E_t\left[\hat{d}^{real}_{t,t+n}\right]\right)^2+E_t\left[\hat{d}^{real}_{t,t+n}\right]E_t\left[\left(\hat{d}^{real}_{t,t+n}\right)^2\right]\right)\\
&\quad-\tfrac{1}{6}\left(E_t\left[\hat{d}^{real}_{t,t+n}\right]\right)^3\\
&=E_t\left[\hat{d}^{real}_{t,t+n}\right]+\tfrac{1}{2}Var_t\left(\hat{d}^{real}_{t,t+n}\right)\\
&\quad+\tfrac{1}{6}\left(E_t\left[\left(\hat{d}^{real}_{t,t+n}\right)^3\right]-\left(E_t\left[\hat{d}^{real}_{t,t+n}\right]\right)^3-3E_t\left[\hat{d}^{real}_{t,t+n}\right]E_t\left[\left(\hat{d}^{real}_{t,t+n}\right)^2\right]\right)
\end{aligned}$$

Now let $\hat{d}^{real}_{t,t+n}=\hat{\lambda}_{t+n}-\hat{\lambda}_t$ and let $\Delta^n\hat{\lambda}_{t+n}\equiv\hat{\lambda}_{t+n}-\hat{\lambda}_t$. Then

$$\begin{aligned}
r^{real}_{t,n}-r^{real}_{ss,n}=-\tfrac{1}{n}\{&E_t\left[\Delta^n\hat{\lambda}_{t+n}\right]+\tfrac{1}{2}Var_t\left(\Delta^n\hat{\lambda}_{t+n}\right)+\tfrac{1}{6}E_t\left[\left(\Delta^n\hat{\lambda}_{t+n}\right)^3\right]\\
&-\tfrac{1}{6}\left(E_t\left[\Delta^n\hat{\lambda}_{t+n}\right]\right)^3-3E_t\left[\Delta^n\hat{\lambda}_{t+n}\right]E_t\left[\left(\Delta^n\hat{\lambda}_{t+n}\right)^2\right]\}
\end{aligned}$$

Similarly, for a nominal bond we have (following some simple algebra)

$$\begin{aligned}
r_{t,n}-r_{ss,n}=-\tfrac{1}{n}\{&E_t\left[\Delta^n\hat{\lambda}_{t+n}-\hat{\Pi}_{t+n}\right]+\tfrac{1}{2}Var_t\left(\Delta^n\hat{\lambda}_{t+n}\right)+\tfrac{1}{2}Var_t\left(\hat{\Pi}_{t+n}\right)-Cov_t\left(\Delta^n\hat{\lambda}_{t+n},\hat{\Pi}_{t+n}\right)\\
&+\tfrac{1}{6}\left(E_t\left[\left(\Delta^n\hat{\lambda}_{t+n}\right)^3\right]-\left(E_t\left[\Delta^n\hat{\lambda}_{t+n}\right]\right)^3-3E_t\left[\Delta^n\hat{\lambda}_{t+n}\right]E_t\left[\left(\Delta^n\hat{\lambda}_{t+n}\right)^2\right]\right)\\
&+\tfrac{1}{6}\left(\left(E_t\left[\hat{\Pi}_{t+n}\right]\right)^3-E_t\left[\left(\hat{\Pi}_{t+n}\right)^3\right]+3E_t\left[\hat{\Pi}_{t+n}\right]E_t\left[\left(\hat{\Pi}_{t+n}\right)^2\right]\right)\\
&+\tfrac{1}{2}Cov_t\left(\left(\hat{\Pi}_{t+n}\right)^2,\Delta^n\hat{\lambda}_{t+n}\right)-\tfrac{1}{2}Cov_t\left(\left(\Delta^n\hat{\lambda}_{t+n}\right)^2,\hat{\Pi}_{t+n}\right)
\end{aligned}$$

$+\frac{1}{2}\left(3E_t\left[\Delta^n\hat{\lambda}_{t+n}\hat{\Pi}_{t+n}\right]-Cov_t\left(\Delta^n\hat{\lambda}_{t+n},\hat{\Pi}_{t+n}\right)\right)\left(E_t\left[\Delta^n\hat{\lambda}_{t+n}\right]-E_t\left[\hat{\Pi}_{t+n}\right]\right)$
}

The difference between the nominal and real interest rates is then given as stated in the text.

B Proof related to inflation risk premia

This section shows that $TP_{t,n}=TP_{t,n}^{real}+TP_{t,n}^{infl}$ for affine Gaussian term structure models and DSGE models approximated up to second order. Simple derivations for real interest rates imply

$$r_{t,n}^{real}=-\frac{1}{n}\left(E_t\left[\sum_{j=1}^n d_{t,t+j}^{real}\right]+\frac{1}{2}V_t\left[\sum_{j=1}^n d_{t,t+j}^{real}\right]\right).$$

For affine Gaussian models this expression holds provided $d_{t,t+j}^{real}\equiv\ln D_{t,t+j}^{real}$ and for DSGE models solved up to second order if $d_{t,t+j}^{real}\equiv\ln\left(D_{t,t+n}^{real}/D_{ss,ss+n}^{real}\right)$. A similar expression holds for nominal interest rates, i.e.

$$r_{t,n}=-\frac{1}{n}\left(E_t\left[\sum_{j=1}^n\left(d_{t,t+j}^{real}-\pi_{t+j}\right)\right]+\frac{1}{2}V_t\left[\sum_{j=1}^n\left(d_{t,t+j}^{real}-\pi_{t+j}\right)\right]\right).$$

The standard decomposition of the real term structure implies

$$\begin{aligned}r_{t,n}^{real}-\frac{1}{n}\sum_{j=1}^n E_t\left[r_{t+j-1,1}^{real}\right] &= -\frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=1+j}^n Cov_t\left(d_{t,t+1+j}^{real},d_{t,t+k}^{real}\right)\\ &\quad -\frac{1}{2n}\sum_{j=1}^n V_t\left[E_{t+j-1}\left[d_{t,t+j}^{real}\right]\right]\end{aligned}$$

and similarly for the nominal term structure, i.e.

$$\begin{aligned}r_{t,n}-\frac{1}{n}\sum_{j=1}^n E_t\left[r_{t+j-1,1}\right] &= -\frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=1+j}^n Cov_t\left(d_{t,t+1+j}^{real}-\pi_{t+1+j},d_{t,t+k}^{real}-\pi_{t+k}\right)\\ &\quad -\frac{1}{2n}\sum_{j=1}^n V_t\left[E_{t+j-1}\left[d_{t,t+j}^{real}-\pi_{t+j}\right]\right]\end{aligned}$$

For these models, real and nominal term premia are given by

$$TP_{t,n}^{real}\equiv-\frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=1+j}^n Cov_t\left(d_{t,t+1+j}^{real},d_{t,t+k}^{real}\right)$$

$$TP_{t,n}\equiv-\frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=1+j}^n Cov_t\left(d_{t,t+1+j}^{real}-\pi_{t+1+j},d_{t,t+k}^{real}-\pi_{t+k}\right)$$

The break-even inflation rates are easily shown to be

$$\begin{aligned}r_{t,n}-r_{t,n}^{real} &= \frac{1}{n}\left(E_t\left[\sum_{j=1}^n\pi_{t+j}\right]-\frac{1}{2}\sum_{j=1}^n V_t\left[\pi_{t+j}\right]\right)\\ &\quad -\frac{1}{n}\left(\sum_{j=0}^{n-1}\sum_{k=1+j}^n Cov_t\left[\pi_{t+1+j},\pi_{t+k}\right]-Cov_t\left[\sum_{j=1}^n d_{t,t+j}^{real},\sum_{j=1}^n\pi_{t+j}\right]\right)\end{aligned}$$

Here, $-\frac{1}{2}\sum_{j=1}^{n} V_t\left[\pi_{t+j}\right]$ is the inflation convexity term, whereas inflation risk premia are given by

$$TP_{t,n}^{infl} \equiv -\frac{1}{n}\left(\sum_{j=0}^{n-1}\sum_{k=1+j}^{n} Cov_t\left[\pi_{t+1+j},\pi_{t+k}\right] - Cov_t\left[\sum_{j=1}^{n} d_{t,t+j}^{real}, \sum_{j=1}^{n}\pi_{t+j}\right]\right).$$

Simple algebra then implies $TP_{t,n} = TP_{t,n}^{real} + TP_{t,n}^{infl}$ as claimed.

C Using the POP method to compute conditional expectations in DSGE models

This section extends the POP method to compute conditional expectations of variables in DSGE models. Consider the case where the model reports a variable denoted r_t and we want to compute conditional expectations for this variable, i.e. $r_t^{(1)} \equiv E_t\left[r_{t+1}\right]$, $r_t^{(2)} \equiv E_t\left[r_{t+2}\right]$, $r_t^{(3)} \equiv E_t\left[r_{t+3}\right]$, etc. The law of iterated expectations implies $r_t^{(2)} = E_t\left[E_{t+1}\left[r_{t+2}\right]\right] = E_t\left[r_{t+1}^{(1)}\right]$ etc. Accordingly, only a formula for $r_t^{(1)} \equiv E_t\left[r_{t+1}\right]$ is required because all other expectations follow by iterating this formula. Below, $r_t^{(1)}$ is denoted by p for simplicity.

Consider the problem

$$T\left(p\left(\mathbf{x}_t,\sigma\right)\right) = E_t\left[T\left(r\left(\mathbf{x}_{t+1},\sigma\right)\right)\right] \tag{36}$$

where σ is the perturbation parameter and $T(\cdot)$ is an invertible and differentiable transformation function. Observe that

$$F\left(\mathbf{x}_t,\sigma\right) \equiv E_t\left[-T\left(p\left(\mathbf{x}_t,\sigma\right)\right) + T\left(r\left(\mathbf{h}\left(\mathbf{x}_t,\sigma\right)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1},\sigma\right)\right)\right] = 0 \tag{37}$$

because

$$\mathbf{x}_{t+1} = \mathbf{h}\left(\mathbf{x}_t,\sigma\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \tag{38}$$

Equation (37) must hold for all values of $(\mathbf{x}_t,\sigma)$, and this allow us to compute all derivatives of p with respect to $(\mathbf{x}_t,\sigma)$ around the deterministic steady state, i.e. $\mathbf{x}_t = \mathbf{x}_{ss}$ and $\sigma = 0$, given derivatives of $\mathbf{h}\left(\mathbf{x}_t,\sigma\right)$ and $r\left(\mathbf{x}_{t+1},\sigma\right)$ around the same point.

For the indices we adopt the convention that the subscript indicates the order of differentiation. I.e. a 1 is for the first time we take derivatives and so on. Thus

$$\alpha_1,\alpha_2,\alpha_3 = 1,2,...n_x \qquad \gamma_1,\gamma_2,\gamma_3 = 1,2,...n_x \qquad \phi_1,\phi_2,\phi_3 = 1,2,...,n_\epsilon$$

where n_x is the number of state variables and n_ϵ is the number of elements in $\boldsymbol{\epsilon}_{t+1}$.

The first order terms:
For $\mathbf{x}_t$:
$\left[F_{\mathbf{x}}\left(\mathbf{x}_{ss},0\right)\right]_{\alpha_1} = E_t\left[-T_p(p)\left[p_{\mathbf{x}}\right]_{\alpha_1} + T_r(r)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\right] = 0$
$\Updownarrow$
$T_p(p)\left[p_{\mathbf{x}}\right]_{\alpha_1} = T_r(r)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}$

Using a log-transformation:
$\mathbf{p_x}(1;:) = \mathbf{r_x}(1,:)\,\mathbf{h_x}$

For σ:
$[F_\sigma(\mathbf{x}_{ss},0)]$
$$= E_t\left[-T_p(p)[p_\sigma] + T_r(r)[r_\mathbf{x}]_{\gamma_1}\left([\mathbf{h}_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) + T_r(r)[r_\sigma]\right] = 0$$
$\Updownarrow$
$[p_\sigma] = 0$
because $E_t\left([\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) = 0$, $[\mathbf{h}_\sigma]^{\gamma_1} = 0$, and $[r_\sigma] = 0$.

The second order terms:
For $(\mathbf{x}_t, \mathbf{x}_t)$:
$$\begin{aligned}[F_{\mathbf{xx}}(\mathbf{x}_{ss},0)]_{\alpha_1\alpha_2} = E_t[&-T_{pp}(p)[p_\mathbf{x}]_{\alpha_1}[p_\mathbf{x}]_{\alpha_2} - T_p(p)[p_{\mathbf{xx}}]_{\alpha_1\alpha_2}\\ &+T_{rr}(r)[r_\mathbf{x}]_{\gamma_1}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}[r_\mathbf{x}]_{\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}\\ &+T_r(r)[r_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}\\ &+T_r(r)[r_\mathbf{x}]_{\gamma_1}[\mathbf{h_{xx}}]^{\gamma_1}_{\alpha_1\alpha_2}] = 0\end{aligned}$$
$\Updownarrow$
$$\begin{aligned}T_p(p)[p_{\mathbf{xx}}]_{\alpha_1\alpha_2} = &-T_{pp}(p)[p_\mathbf{x}]_{\alpha_1}[p_\mathbf{x}]_{\alpha_2}\\ &+T_{rr}(r)[r_\mathbf{x}]_{\gamma_1}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}[r_\mathbf{x}]_{\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}\\ &+T_r(r)[r_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}\\ &+T_r(r)[r_\mathbf{x}]_{\gamma_1}[\mathbf{h_{xx}}]^{\gamma_1}_{\alpha_1\alpha_2}\end{aligned}$$
Using a log-transformation:
$\mathbf{p_{xx}} = \mathbf{h_x'}\mathbf{r_{xx}}\mathbf{h_x} + \sum_{\gamma_1=1}^{n_x}\mathbf{r_x}(1,\gamma_1)\,\mathbf{h_{xx}}(\gamma_1,:,:)$

For (σ,σ):
$$\begin{aligned}[F_{\sigma\sigma}(\mathbf{x}_{ss},0)] = E_t[&-T_{pp}(p)[p_\sigma][p_\sigma] - T_p(p)[p_{\sigma\sigma}]\\ &+T_{rr}(r)\left([r_\mathbf{x}]_{\gamma_2}\left([\mathbf{h}_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) + [r_\sigma]\right)[r_\mathbf{x}]_{\gamma_1}\left([\mathbf{h}_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\ &+T_r(r)\left([r_{\mathbf{xx}}]_{\gamma_1\gamma_2}\left([\mathbf{h}_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) + [r_{\mathbf{x}\sigma}]_{\gamma_1}\right)\left([\mathbf{h}_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\ &+T_r(r)[r_\mathbf{x}]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}\\ &+T_{rr}(r)\left([r_\mathbf{x}]_{\gamma_2}\left([\mathbf{h}_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) + [r_\sigma]\right)[r_\sigma]\\ &+T_r(r)[r_{\sigma\sigma}]] = 0\end{aligned}$$
$\Updownarrow$
$$\begin{aligned}T_p(p)[p_{\sigma\sigma}] = &T_{rr}(r)[r_\mathbf{x}]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[r_\mathbf{x}]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}\\ &+T_r\left(r^{t+1}\right)[r_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}\\ &+T_r(r)[r_\mathbf{x}]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}\\ &+T_r(r)[r_{\sigma\sigma}]\end{aligned}$$
because $[\mathbf{h}_\sigma]^{\gamma_1} = 0$ and $[r_{\mathbf{x}\sigma}]_{\gamma_1} = 0$. Using a log-transformation:
$p_{\sigma\sigma} = \mathbf{r_x}(1,:)\,\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{r_x}(1,:)' + trace(\boldsymbol{\eta}'\mathbf{r_{xx}}\boldsymbol{\eta}) + \mathbf{r_x}(1,:)\,\mathbf{h}_{\sigma\sigma} + r_{\sigma\sigma}$

The second-order term $p_{\mathbf{x}\sigma}$ is known to be zero (see Schmitt-Grohé & Uribe (2004)).

Third order terms:
For $(\mathbf{x}_t, \mathbf{x}_t, \mathbf{x}_t)$:

$$T_p(p)[p_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3} = -T_{ppp}(p)[p_{\mathbf{x}}]_{\alpha_1}[p_{\mathbf{x}}]_{\alpha_2}[p_{\mathbf{x}}]_{\alpha_3} - T_{pp}(p)[p_{\mathbf{x}}]_{\alpha_1\alpha_3}[p_{\mathbf{x}}]_{\alpha_2}$$
$$-T_{pp}(p)[p_{\mathbf{x}}]_{\alpha_1}[p_{\mathbf{xx}}]_{\alpha_2\alpha_3} - T_{pp}(p)[p_{\mathbf{xx}}]_{\alpha_1\alpha_2}[p_{\mathbf{x}}]_{\alpha_3}$$
$$+T_{rrr}(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}[r_{\mathbf{x}}]_{\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}[r_{\mathbf{x}}]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}$$
$$+T_{rr}(r)[r_{\mathbf{xx}}]_{\gamma_1\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}[r_{\mathbf{x}}]_{\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}$$
$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h_{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}[r_{\mathbf{x}}]_{\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}$$
$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}[r_{\mathbf{xx}}]_{\gamma_2\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}$$
$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}[r_{\mathbf{x}}]_{\gamma_2}[\mathbf{h_{xx}}]^{\gamma_2}_{\alpha_2\alpha_3}$$
$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[r_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}$$
$$+T_r(r)[r_{\mathbf{xxx}}]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}$$
$$+T_r(r)[r_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h_{xx}}]^{\gamma_2}_{\alpha_2\alpha_3}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}$$
$$+T_r(r)[r_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}[\mathbf{h_{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}$$
$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h_{xx}}]^{\gamma_1}_{\alpha_1\alpha_2}$$
$$+T_r(r)[r_{\mathbf{xx}}]_{\gamma_1\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$$
$$+T_r(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h_{xxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}$$

Using a log-transformation:

$$\mathbf{p_{xxx}}(\alpha_1,\alpha_2,\alpha_3) = \sum_{\gamma_3=1}^{n_x}\mathbf{h_x}(:,\alpha_1)'\,\mathbf{r_{xxx}}(:,:,\gamma_3)\,\mathbf{h_x}(:,\alpha_2)\,\mathbf{h_x}(\gamma_3,\alpha_3)$$
$$+\mathbf{h_x}(:,\alpha_1)'\,\mathbf{r_{xx}}\mathbf{h_{xx}}(:,\alpha_2,\alpha_3)$$
$$+\sum_{\gamma_1=1}^{n_x}\mathbf{r_{xx}}(\gamma_1,:)\,\mathbf{h_x}(:,\alpha_2)\,\mathbf{h_{xx}}(\gamma_1,\alpha_1,\alpha_3)$$
$$+\sum_{\gamma_1=1}^{n_x}\mathbf{r_{xx}}(\gamma_1,:)\,\mathbf{h_x}(:,\alpha_3)\,\mathbf{h_{xx}}(\gamma_1,\alpha_1,\alpha_2)$$
$$+\mathbf{r_x}(1,:)\,\mathbf{h_{xxx}}(:,\alpha_1,\alpha_2,\alpha_3)$$

For $\left(\sigma^2, \mathbf{x}_t\right)$:

$$T_p(p)[p_{\sigma\sigma\mathbf{x}}]_{\alpha_3} = -T_{pp}(p)[p_{\mathbf{x}}]_{\alpha_3}[p_{\sigma\sigma}]$$
$$+T_{rrr}(r)[r_{\mathbf{x}}]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[r_{\mathbf{x}}]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[r_{\mathbf{x}}]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+3T_{rr}(r)[r_{\mathbf{xx}}]_{\gamma_2\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[r_{\mathbf{x}}]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+T_r(r)[r_{\mathbf{xxx}}]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$
$$+T_r(r)[r_{\mathbf{xx}}]_{\gamma_1\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$
$$+T_r(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\sigma\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}$$
$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[r_{\sigma\sigma}]$$
$$+T_r(r)[r_{\sigma\sigma\mathbf{x}}][\mathbf{h_x}]^{\gamma_3}_{\alpha_3}$$

Using a log-transformation:

$$\mathbf{p}_{\sigma\sigma\mathbf{x}} = 2\mathbf{r_x}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{r_{xx}}\mathbf{h_x} + \sum_{\gamma_3=1}^{n_x} trace\left(\boldsymbol{\eta}'\mathbf{r_{xxx}}(:,:,\gamma_3)\,\boldsymbol{\eta}\right)\mathbf{h_x}(\gamma_3,:)$$
$$+\mathbf{h}'_{\sigma\sigma}\mathbf{r_{xx}}\mathbf{h_x} + \mathbf{r_x}\mathbf{h}_{\sigma\sigma\mathbf{x}} + \mathbf{r}_{\sigma\sigma\mathbf{x}}\mathbf{h_x}$$

For $\left(\sigma^3\right)$:

$$T_p(p)[p_{\sigma\sigma\sigma}] =$$
$$T_{rrr}(r)[r_{\mathbf{x}}]_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[r_{\mathbf{x}}]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[r_{\mathbf{x}}]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3T_{rr}(r)[r_{\mathbf{xx}}]_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[r_{\mathbf{x}}]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+T_r(r)[r_{\mathbf{xxx}}]_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$

$+T_r\left(r\right)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\sigma}\right]^{\gamma_1}$
$+T_r\left(r\right)\left[r_{\sigma\sigma\sigma}\right]$

Here we introduce the additional notation

$$\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} = \begin{cases} m^3\left(\boldsymbol{\epsilon}_{t+1}\left(\phi_1, 1\right)\right) & \text{if } \phi_1 = \phi_2 = \phi_3 \\ 0 & \text{else} \end{cases}$$

where $m^3\left(\boldsymbol{\epsilon}_{t+1}\left(\phi_1, 1\right)\right)$ denotes the third moment of $\boldsymbol{\epsilon}_{t+1}\left(\phi_1, 1\right)$. Using a log-transformation we get

$\left[p^t_{\sigma\sigma\sigma}\right] = \sum_{\phi_2=1}^{n_e}\sum_{\phi_3=1}^{n_e}\mathbf{r}_{\mathbf{x}}\boldsymbol{\eta}\left(:,\phi_3\right)\mathbf{r}_{\mathbf{x}}\boldsymbol{\eta}\left(:,\phi_2\right)\mathbf{r}_{\mathbf{x}}\boldsymbol{\eta}\mathbf{m}^3\left(:,\phi_2,\phi_3\right)$
$+3\sum_{\phi_2=1}^{n_e}\sum_{\phi_3=1}^{n_e}\boldsymbol{\eta}\left(:,\phi_2\right)'\mathbf{r}_{\mathbf{xx}}\boldsymbol{\eta}\left(:,\phi_3\right)\mathbf{r}_{\mathbf{x}}\boldsymbol{\eta}\mathbf{m}^3\left(:,\phi_2,\phi_3\right)$
$+\sum_{\gamma_1=1}^{n_x}\sum_{\phi_2=1}^{n_e}\sum_{\phi_3=1}^{n_e}\boldsymbol{\eta}\left(:,\phi_2\right)'\mathbf{r}_{\mathbf{xxx}}\left(\gamma_1,:,:\right)\boldsymbol{\eta}\left(:,\phi_3\right)\boldsymbol{\eta}\left(\gamma,:\right)\mathbf{m}^3\left(:,\phi_2,\phi_3\right)$
$+\mathbf{r}^{t+1}_{\mathbf{x}}\mathbf{h}^t_{\sigma\sigma\sigma}$
$+r^{t+1}_{\sigma\sigma\sigma}$

The third-order term $p_{\mathbf{xx}\sigma}$ is known to be zero (see Andreasen (forthcoming)).

Table 1: Estimation results

The standard errors are computed from the variance of the score function which is pre- and post multiplied by the inverse of the Hessian matrix. Given these estimates, the non-linear calibration implies $\pi_{ss} = 1.0157$.

Label		Estimate	SE
Discount factor	β	0.9992	0.0002
Habit formation	b	0.2876	0.0398
Preference parameter for weight on leisure	ξ_1	0.6289	0.0475
Preference parameter for curvature in utility index	ξ_2	2.3931	0.2567
Preference parameter for Epstein-Zin-Weil specification	ξ_3	-183.1	24.77
Adjustment costs in investment	κ	5.4033	0.3137
Price stickiness	α	0.7925	0.0083
Interest rate rule: degree of interest smoothing	ρ_r	0.8838	0.0115
Interest rate rule: inflation gap	β_π	1.4895	0.0820
Interest rate rule: output gap	β_y	0.0505	0.0104
Non-stationary technology shocks	ρ_z	0.5289	0.0814
Preference shocks	ρ_d	0.9635	0.0023
Firms' fixed costs	ρ_ψ	0.9969	0.0016
Stationary technological. shocks	ρ_a	0.8450	0.0149
Investment shocks	ρ_e	0.9870	0.0013
The central bank's inflation reaction	ρ_{β_π}	0.9711	0.0046
Std. for non-stationary technology shocks	σ_z	0.0053	0.0006
Std. for preference shocks	σ_d	0.0227	0.0039
Std. for shocks to firms' fixed costs	σ_ψ	0.0198	0.0038
Std. for stationary technological. shocks	σ_a	0.0097	0.0017
Std. for investment shocks	σ_e	0.0203	0.0017
Std. for the central bank's inflation reaction	σ_{β_π}	0.0280	0.0053
Std. for shocks to the interest rate rule	σ_{ϵ_r}	0.0019	0.0003
Std. for measurement errors in 4-quarters inflation expectations	$\sigma_{\pi^e_4}$	0.0041	0.0005
Std. for measurement errors in 12-quarters inflation expectations	$\sigma_{\pi^e_{12}}$	0.0033	0.0005
Std. for measurement errors in long inflation expectations	$\sigma_{\pi^e_{long}}$	0.0010	0.0002
Parameter for measurement error in nominal yield curve	γ_1	-4.7867	0.1280
Parameter for measurement error in nominal yield curve	γ_2	-0.1835	0.0087
Parameter for measurement error in nominal yield curve	γ_3	0.0033	0.0002
Parameter for measurement error in real yield curve	γ_4	1.4051	0.0674

Table 2: Variance decomposition

Summing the contributions of the various shocks might not equal 100 due to rounding errors.

	Non-stationary tech. shocks	Preference shocks	Shocks to firms' fixed costs	Stationary tech. shocks	Investment shocks	Shocks to CB's inflation raction	Interest rate shocks	Measure-ment errors	Total
1-step									100
$r_{t,1}$	2	4	0	4	0	0	35	55	100
$r_{t,4}$	8	20	0	14	1	2	20	35	100
$r_{t,12}$	20	48	0	15	1	3	5	7	100
$r_{t,20}$	25	57	0	10	1	3	2	2	100
$r_{t,28}$	27	60	0	7	1	2	1	2	100
$r_{t,40}$	28	59	1	4	1	1	1	6	100
$r_{t,12}^{real}$	3	1	1	3	1	2	62	27	100
$r_{t,20}^{real}$	1	2	5	7	1	2	61	21	100
$r_{t,28}^{real}$	0	9	8	8	1	2	50	22	100
$r_{t,40}^{real}$	1	12	6	4	0	1	22	54	100
π_t	13	27	0	26	1	3	29	1	100
$\triangle c_t$	30	17	5	1	10	4	33	0	100
$\triangle i_t$	47	1	1	0	35	1	11	4	100
$\triangle y_t$	44	9	2	0	11	3	30	0	100
$\pi_{t,4}^e$	17	31	0	3	2	9	7	31	100
$\pi_{t,12}^e$	18	26	0	0	2	5	0	49	100
$\pi_{t,long}^e$	1	1	1	0	21	0	0	76	100
8-steps									
$r_{t,1}$	1	42	0	14	1	4	15	23	100
$r_{t,4}$	2	66	0	16	2	6	3	4	100
$r_{t,12}$	2	79	0	9	3	6	0	0	100
$r_{t,20}$	3	85	0	5	2	5	0	0	100
$r_{t,28}$	3	88	0	3	2	3	0	0	100
$r_{t,40}$	3	89	1	2	1	2	0	0	100
$r_{t,12}^{real}$	2	10	16	9	9	23	22	9	100
$r_{t,20}^{real}$	1	11	37	15	6	13	14	4	100
$r_{t,28}^{real}$	0	31	38	10	3	8	7	3	100
$r_{t,40}^{real}$	0	45	32	5	2	5	3	8	100
π_t	2	62	0	22	2	9	3	0	100
$\triangle c_t$	6	41	14	1	27	10	3	0	100
$\triangle i_t$	9	2	3	0	84	1	1	0	100
$\triangle y_t$	10	29	6	0	42	9	3	0	100
$\pi_{t,4}^e$	3	65	0	2	6	21	1	2	100
$\pi_{t,12}^e$	4	68	1	0	6	16	0	5	100
$\pi_{t,long}^e$	0	3	6	0	77	1	0	13	100

Figure 1: Historical model fit

The historical time series are denoted by black lines and the model-implied series evaluated at the estimated states are denoted by red lines. The numbers in parenthesises are the correlation between the historical series and the model-implied series.

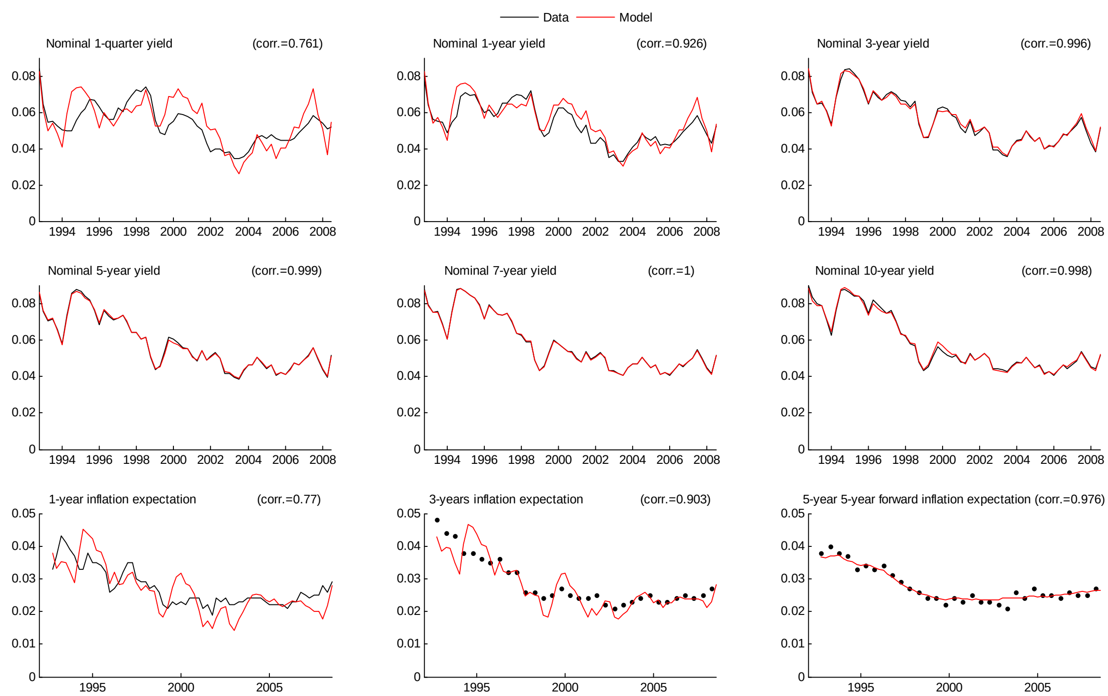

Figure 1: Continued

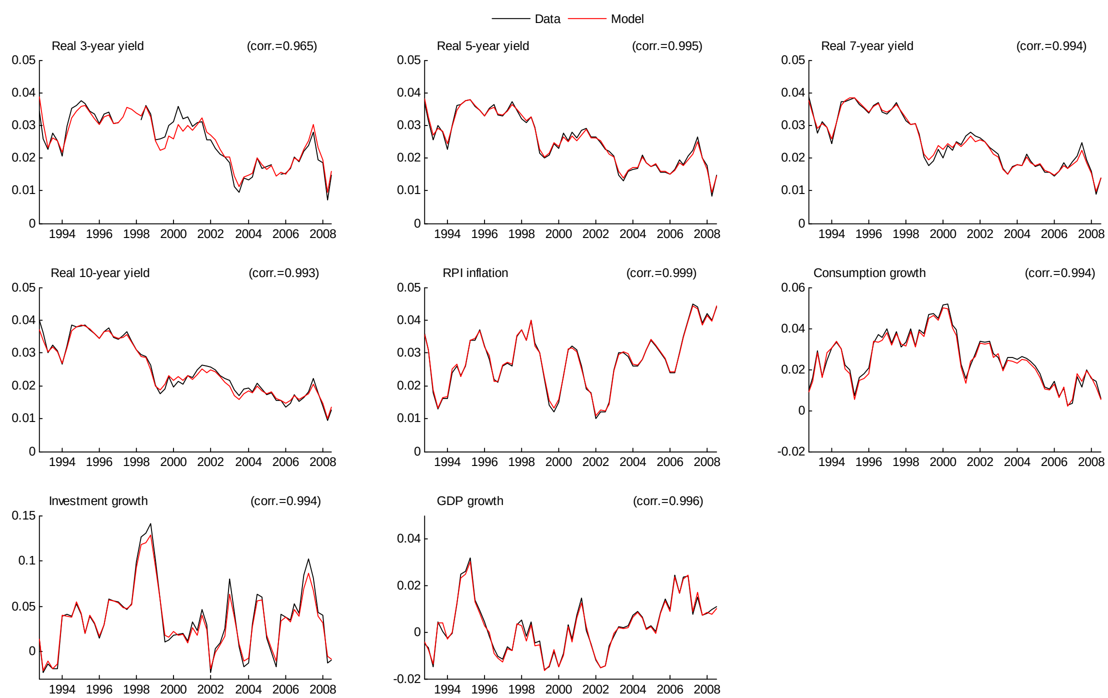

Figure 2: Impulse response functions

Each row shows responses to a positive one-standard deviation shock for a number of variables. All responses are in deviation from the steady state, except for term premia which are reported in annualized basis points in deviation from their mean values. The order of the structural shocks is: shock to firms' fixed costs (ψ_t), non-stationary technology shock (z_t), stationary technology shock (a_t), preference shock (d_t), investment shock (e_t), shock to the central bank's inflation reaction ($\beta_{\pi,t}$), and monetary policy shock ($\epsilon_{r,t}$).

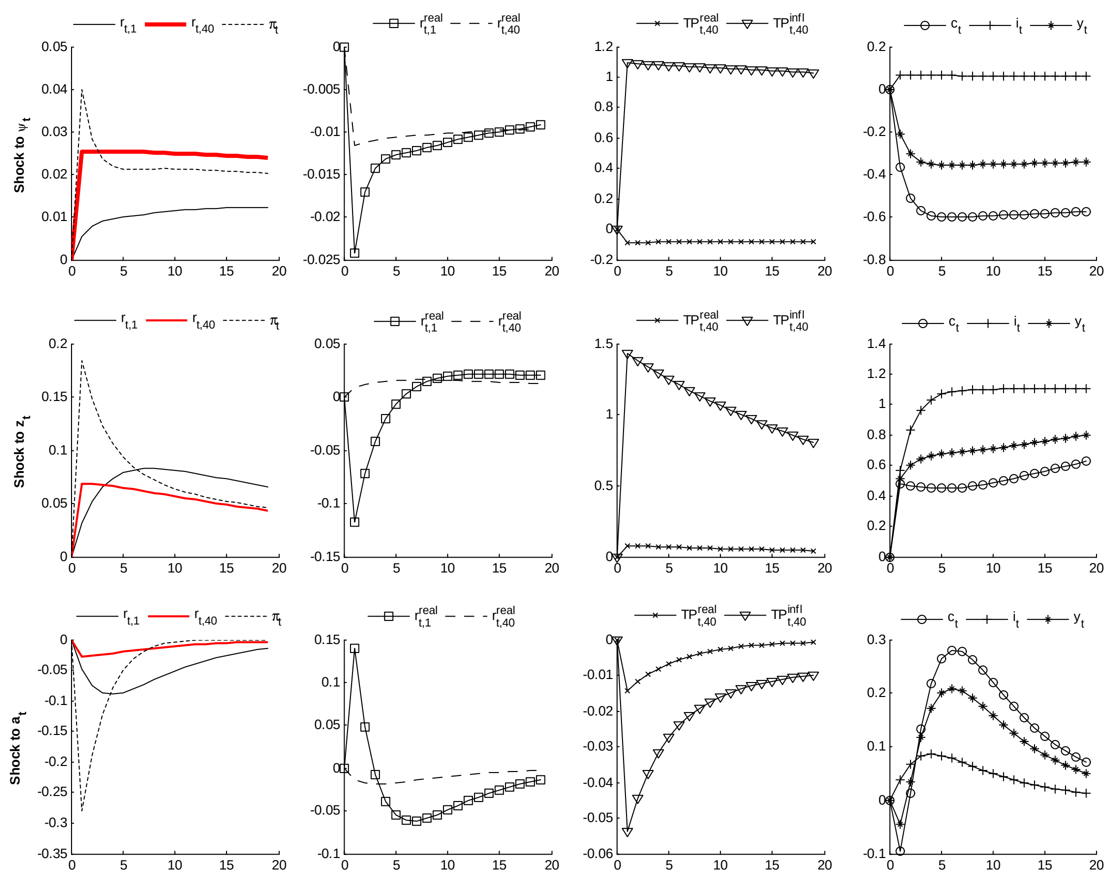

Figure 2: Continued

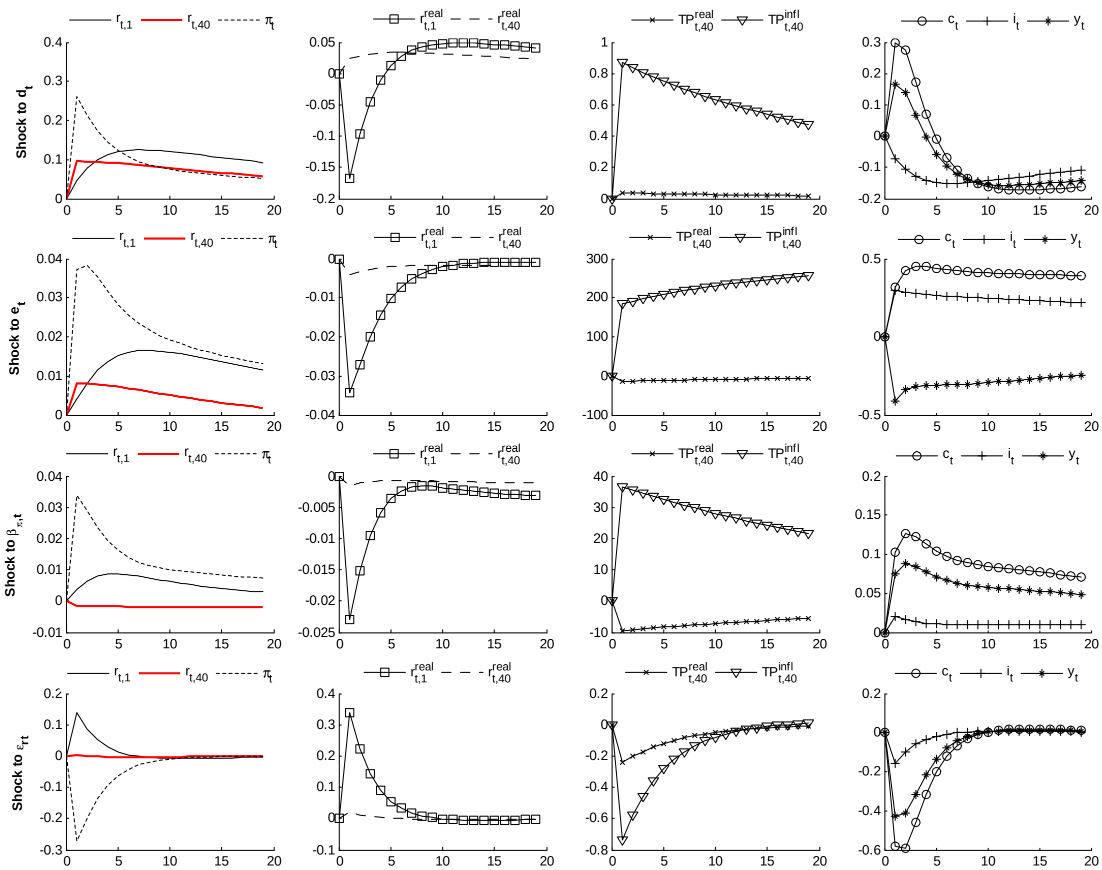

Figure 3: Historical time series for term premia

Figure 4: Structural decomposition of the 10-year inflation risk premium

All counter-factuals are shown in annualized basis points.

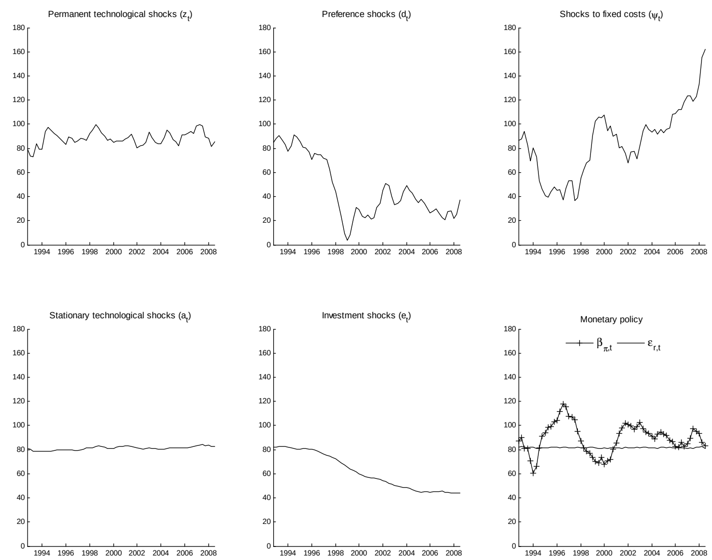

Figure 5: Estimated structural shocks

All shocks are shown in percentage deviation from steady state.

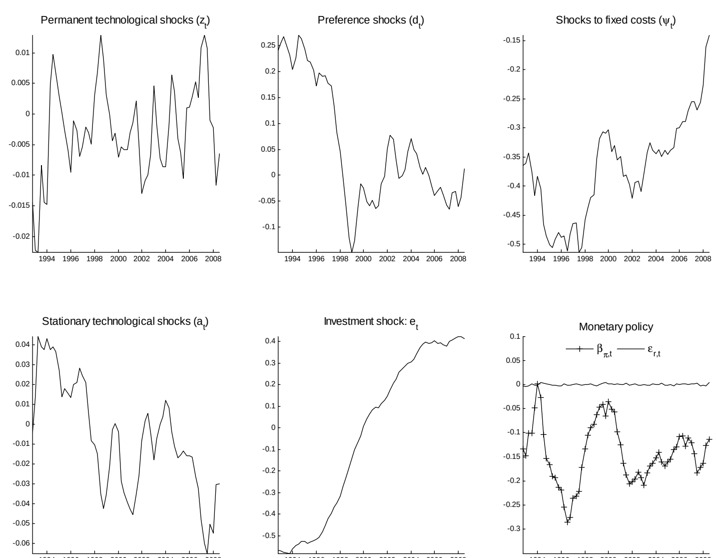

Figure 6: The market price of nominal risk and the quantity of inflation risk